Advanced Condensed Matter 2 - Problem Set 1

In a convectional metal at $T = 0$ K, the ground state is obtained by filling electrons in all the available states with $k < k_F$, k_F being the Fermi wave vector equivalent to that of the highest occupied electronic state. We are going to consider what happens if we add two electrons to this Fermi "sea", that are allowed to attract each other. The general form of the Schroedinger equation for the additional electrons is given by

$$\left[\frac{\mathbf{p}_1^2}{2m} + \frac{\mathbf{p}_2^2}{2m} + U(\mathbf{r}_1, \mathbf{r}_2)\right] \psi\left(\mathbf{r}_1\sigma_1, \mathbf{r}_2\sigma_2\right) = E\psi\left(\mathbf{r}_1\sigma_1, \mathbf{r}_2\sigma_2\right), \tag{1}$$

where $\mathbf{r}$ and σ refer to the position and spin coordinates and the 1 and 2 label the electrons. In this problem you will first construct ψ and then use it to solve the Schroedinger equation to obtain the binding energy of the additional electron pair.

1. Taking a basis set comprised of the product of plane waves of the form $\phi_{\mathbf{k}} = \left(1/\sqrt{V}\right)\exp\left(i\mathbf{k}\cdot\mathbf{r}\right)$ and spin functions labeled α (spin up) and β (spin down), construct the four possible Slater determinant functions between $\mathbf{k}\alpha$, $-\mathbf{k}\alpha$, $\mathbf{k}\beta$, and $-\mathbf{k}\beta$ basis functions. *Hint: the* $\mathbf{k}\alpha$, $-\mathbf{k}\alpha$ *slater determinant of the would be obtained by calculating*

$$\Phi_1 = \frac{1}{\sqrt{2}}\begin{vmatrix} \phi_{\mathbf{k}\alpha}(1) & \phi_{\mathbf{k}\alpha}(2) \\ \phi_{-\mathbf{k}\alpha}(1) & \phi_{-\mathbf{k}\alpha}(2) \end{vmatrix}. \tag{2}$$

2. Using the slater determinant functions obtained in part 1, construct the singlet ($S = 0$) and triplet ($S = 1$) state functions as superpositions. Use the fact that the spatial portion of the singlet function is symmetric under particle exchange, while the spin function is antisymmetric (and vice versa for the triplet state).

3. We will now use the singlet state function you obtained in part 2. The most general $S = 0$ pair wave function is obtained by taking an arbitrary weighted superposition of the singlet state functions over all $\mathbf{k}$. Let us call the weighting factor $g(\mathbf{k})$. For the remainder of this problem, we will assume that $g(\mathbf{k}) = 0$ for $E_{\mathbf{k}} < E_F$ and $E_{\mathbf{k}} > E_F + \hbar\omega_D$ (the former condition is a statement of the Pauli exclusion principle and the latter implying that the attractive potential is mediated by phonons). Using the generalized $S = 0$ function and equation (1), show that the transformed Schroedinger equation is given by

$$\left[2E_{\mathbf{k}} - E\right] g(\mathbf{k}) + \sum_{\mathbf{k}'} U_{\mathbf{k}\mathbf{k}'} g(\mathbf{k}') = 0, \tag{3}$$

   where $U_{\mathbf{k}\mathbf{k}'}$ is the appropriate matrix element of the attractive potential $U(\mathbf{r}_1, \mathbf{r}_2)$. As an aside, comment on the symmetry of $g(\mathbf{k})$ (i.e. its odd or even-ness for singlet and triplet states, respectively).

4. Solve equation (3) for $E = E_{pair}$ making the simplifying assumptions that $U_{\mathbf{k}\mathbf{k}'}$ is a constant. To complete this part, you will need convert a sum to an integral over a function multiplied by the density of states. To make this even simpler, assume that the density of states is constant over the bounds of integration. Finally, solve for the pair binding energy $\Delta_b = 2E_F - E_{pair}$.