Last time

- Insulators, metals, etc. (finish from last time)
- Reflection/Transmission
- Interference: Fabry – Perot resonator
- Wave equation
- Optical Functions
- Symmetry

FOX Ch. 2
Fox 5.4.2 Diode Laser
Fox 11.2.2 Resonant nonlinearities
Fox 11.4.3 Resonant nonlinearities in semiconductors
Fox App.A Electromagnetism in dielectrics

Today

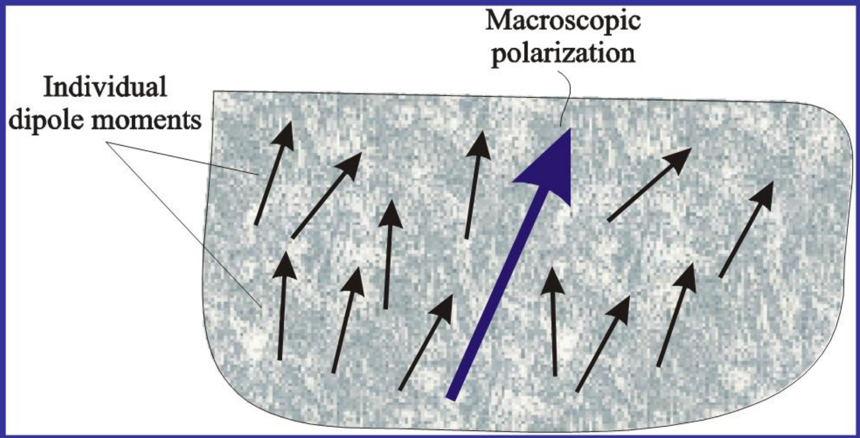
- Lorentz model (Fox 2)
- Kramers kronig (Fox 2.2.5)
    - **Ch. 7.3 in Mathemetical methods for physicists**, George Arfken, Academic press 1985. ISBN 0 12 059810 8
- Raman effect (Fox 10.5)
- Metals and Doped Semiconductors (Fox 7)
    - Drude model
    - Interband transitions
    - Doping levels in semiconductors
    - Plasmon excitations

Response: models

Optical Properties of Gases

1 mole of ideal gas: 22.4 dm^3 (1 atm, 0^oC) ➔ Distance ~10 nm >> size of molecules

$E_{dipole} \propto r^{-3}$ - no interactions



Let us assume no permanent dipoles

Induced dipole moment per molecule: $\mathbf{p} = \alpha \mathbf{E}$

Total moment: $\mathbf{P}_{total} = \mathbf{E} \sum_{i=1}^{N} \alpha_i = N\alpha \mathbf{E}$

Polarization: $\mathbf{P} = \dfrac{N\alpha \mathbf{E}}{V}$

As $\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$ the electric susceptibility: $\chi_e = \dfrac{N\alpha}{\varepsilon_0 V}$

N molecules
with polarizability α
in volume *V*

The relative permittivity: $\varepsilon_r = 1 + \chi_e = 1 + \dfrac{N\alpha}{\varepsilon_0 V}$

Refractive index: $n = \sqrt{1 + \dfrac{N\alpha}{\varepsilon_0 V}} \cong 1 + \dfrac{N\alpha}{2\varepsilon_0 V}$ - diluted gas

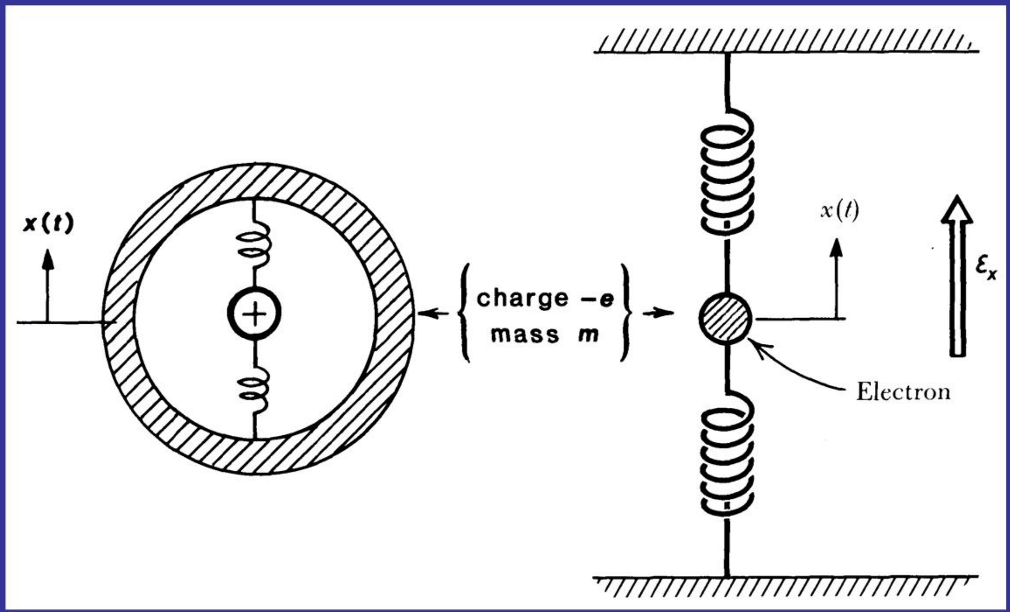
For ideal gases, $pV = Nk_B T$, or $\dfrac{N}{V} = \dfrac{p}{k_B T}$ $n = 1 + \dfrac{\alpha\, p}{2\varepsilon_0 k_B T}; \;\; \alpha = 2\varepsilon_0 k_B (n-1)\dfrac{T}{p}$

Damped harmonic oscillator

Harmonic oscillator model

Harmonic potential: $V(t) = \frac{k}{2} x^2(t)$

Force: $F = -eE(t)$



Equation of motion:

$$\frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = -\frac{e}{m} E(t)$$

$\omega_0^2 = \frac{k}{m}$ - frequency

γ - decay (‘friction’)

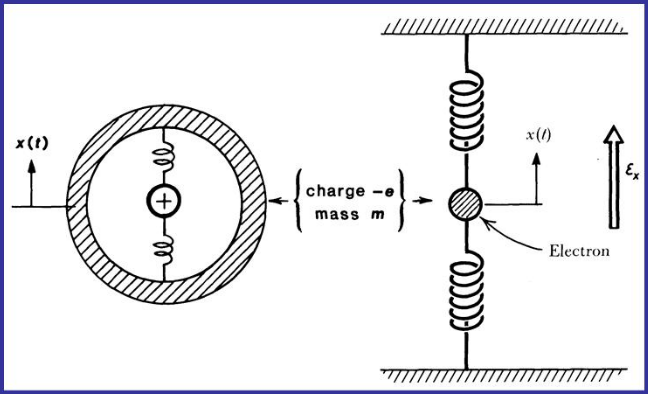
$E(t) = E_0 e^{-i\omega t}$ - monochromatic electric field

Harmonic oscillator

$$\frac{d^2x}{dt^2}+\gamma\frac{dx}{dt}+\omega_0^2 x=-\frac{e}{m}E_0 e^{-i\omega t} \rightarrow \left(-\omega^2-i\gamma\omega+\omega_0^2\right)x_0=-\frac{e}{m}E_0$$

Ansatz solution: $x(t)=x_0 e^{-i\omega t}$

$$x_0=-\frac{e}{m}\frac{E_0}{\left(\omega_0^2-\omega^2-i\gamma\omega\right)}$$



Induced dipole moment: $p(t)=ex(t)=-\frac{e^2}{m}\frac{E_0 e^{-i\omega t}}{\omega_0^2-\omega^2-i\gamma\omega}$

Susceptibility: $\chi(\omega)=\frac{p(t)}{\varepsilon_0 E(t)}=\frac{e^2}{\varepsilon_0 m}\frac{1}{\omega_0^2-\omega^2-i\gamma\omega}$

$$\chi(\omega)=\chi'(\omega)+i\chi''(\omega)$$

$$\chi'(\omega)=\frac{e^2}{\varepsilon_0 m}\frac{\omega_0^2-\omega^2}{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2} \qquad \chi''(\omega)=\frac{e^2}{\varepsilon_0 m}\frac{\gamma\omega}{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}$$

Harmonic oscillator

$$\chi(\omega) = \frac{e^2}{\varepsilon_0 m} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

Assumptions

- Restoring force linear in x ("spring")
- Damping linear in dx/dt
- Force along field
  (or: dipole along field isotropic)
- Single dipole
  (or: field of other dipoles neglected)

N identical oscillators:

$$\chi(\omega) = \frac{Ne^2}{\varepsilon_0 m} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} = \frac{{\omega_p}^2}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

Oscillator strength $\quad {\omega_p}^2 = \frac{Ne^2}{\varepsilon_0 m}$

Different oscillators

$$\chi(\omega) = \sum_j \frac{f_j {\omega_{pj}}^2}{\omega_{0j}^2 - \omega^2 - i\gamma_j \omega}$$

f_j : oscillator strength

In the vicinity of ω_0:

$$\left(\omega_0^2 - \omega^2\right) = (\omega_0 + \omega)(\omega_0 - \omega) \approx 2\omega(\omega_0 - \omega)$$

$$\chi(\omega) \approx \frac{e^2}{2\varepsilon_0 m \omega_0} \frac{1}{\omega_0 - \omega - i\gamma/2}$$

$$\chi'(\omega) \approx \frac{e^2}{2\varepsilon_0 m \omega_0} \frac{\omega_0 - \omega}{(\omega_0 - \omega)^2 + (\gamma/2)^2}$$

$$\chi''(\omega) \approx \frac{e^2}{2\varepsilon_0 m \omega_0} \frac{\gamma/2}{(\omega_0 - \omega)^2 + (\gamma/2)^2}$$

Harmonic oscillator

$$\chi'(\omega) \approx \frac{e^2}{2\varepsilon_0 m\omega_0} \frac{\omega_0 - \omega}{(\omega_0 - \omega)^2 + (\gamma/2)^2}$$

$$\chi''(\omega) \approx \frac{e^2}{2\varepsilon_0 m\omega_0} \frac{\gamma/2}{(\omega_0 - \omega)^2 + (\gamma/2)^2}$$

Imaginary part:
Full Width Half Maximum (FWHM)

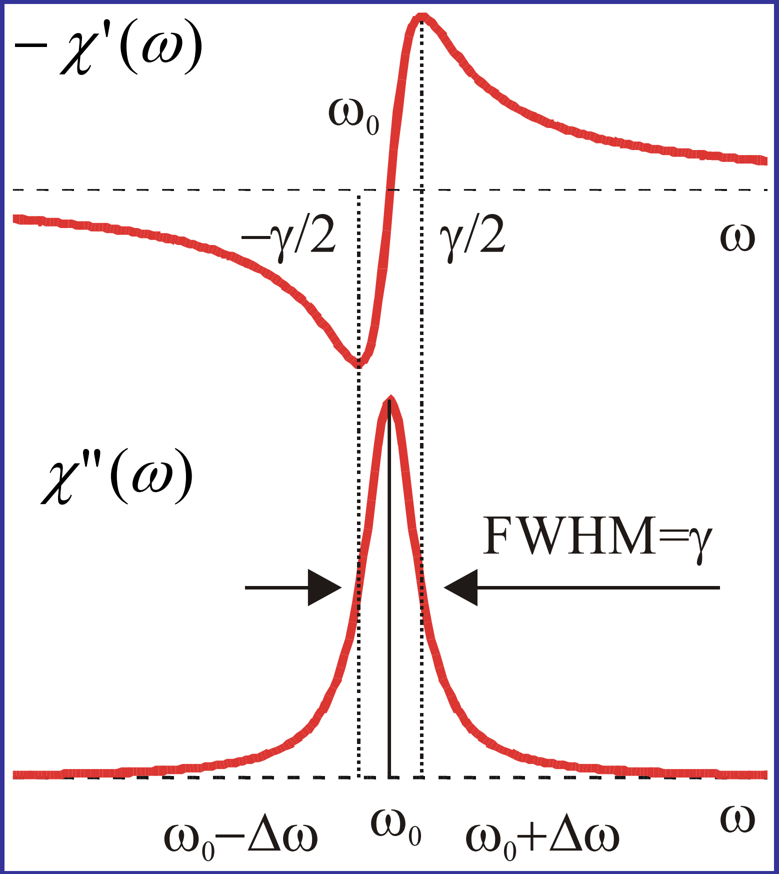
$$\frac{1}{2(\gamma/2)} = \frac{\gamma/2}{(\Delta\omega)^2 + (\gamma/2)^2}$$

$$\Delta\omega = \pm\gamma/2; \quad \mathrm{FWHM} = \gamma$$

Real part:
positions of extrema

$$(\Delta\omega)^2 + (\gamma/2)^2 - 2(\Delta\omega)^2 = 0$$

$$\Delta\omega = \pm\gamma/2$$



Optical functions

Once the dielectric susceptibility is known, we may derive other optical functions

Dielectric function $\varepsilon_r(\omega) = 1 + \chi_e(\omega)$

Refractive index $n(\omega) = \sqrt{\varepsilon_r(\omega)}$

Optical conductivity $\sigma(\omega) = -i\omega\varepsilon_0\chi_e(\omega)$

Wavevector $k(\omega) = n(\omega)\frac{\omega}{c} = n(\omega)k_{vac}$

Also:

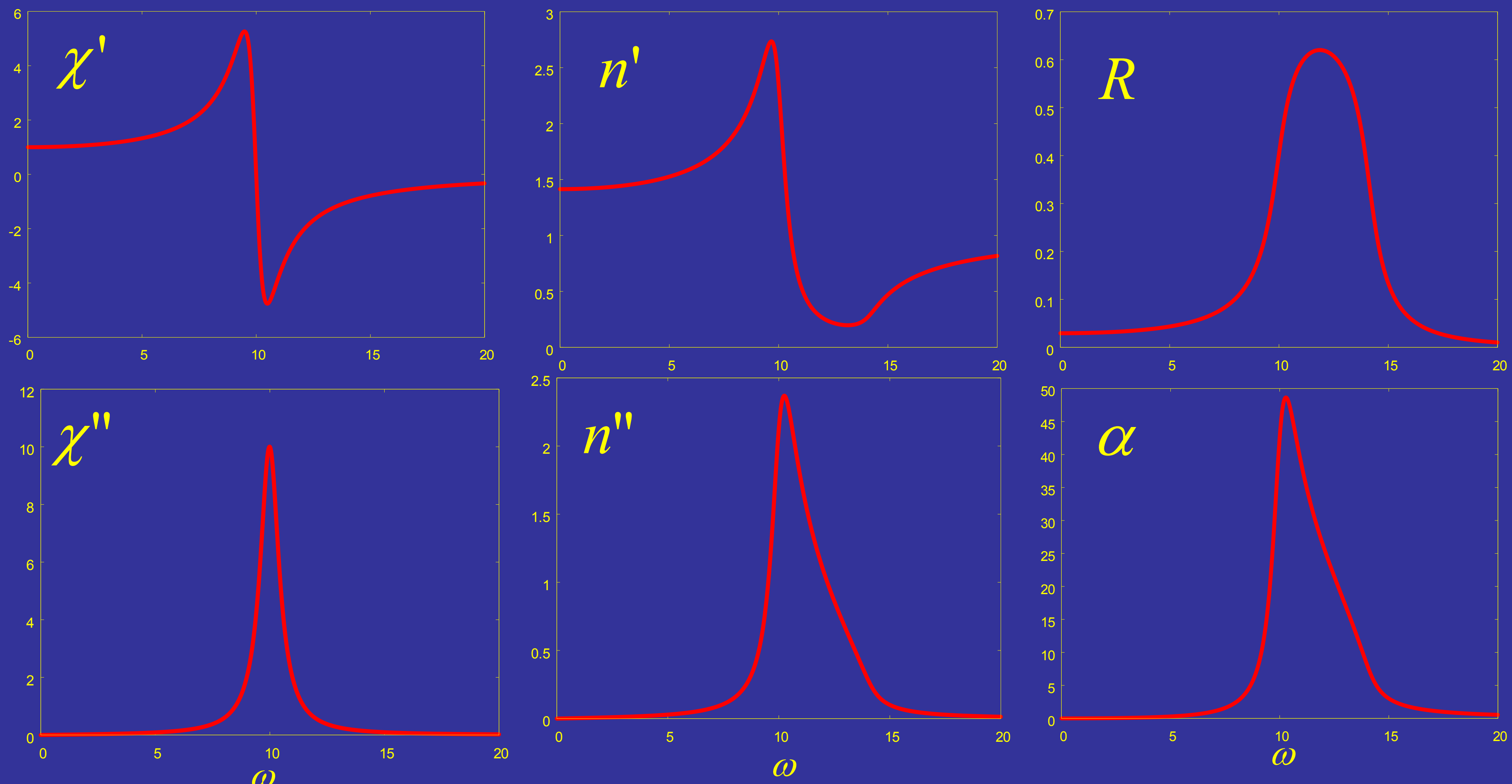
reflection at perpendicular incidence: $R(\omega) = \frac{|n(\omega)-1|^2}{|n(\omega)+1|^2}$

absorption coefficient: $\alpha(\omega) = 2\frac{\omega}{c}n''(\omega)$

Optical functions

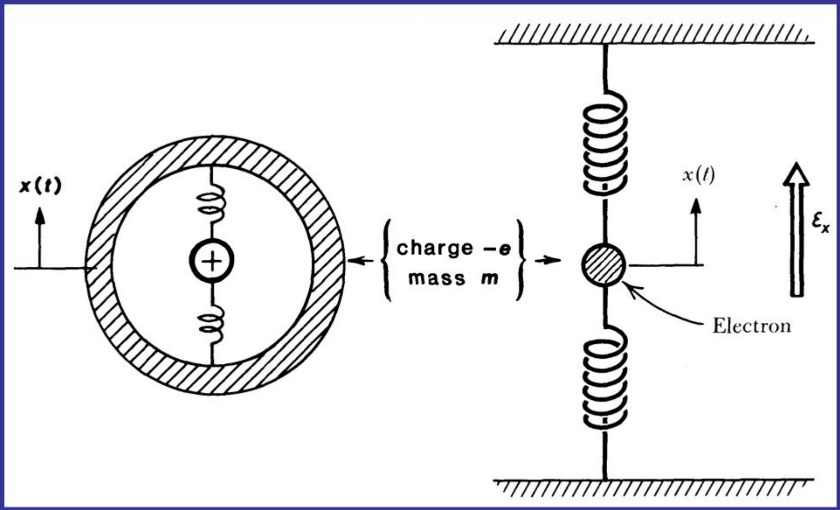
$$\chi(\omega) = \frac{\omega_p^{\ 2}}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

$$\omega_0 = \omega_p = 10; \quad \gamma = 1$$



Lorentz model

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = -\frac{e}{m} E(t)$$



$\omega_0^2 = \frac{k}{m}$ - frequency

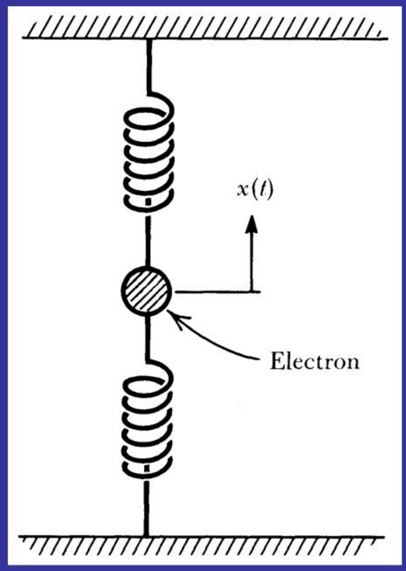
γ - decay ('friction')

$E(t) = E_0 e^{-i\omega t}$ - monochromatic electric field

$$\chi(\omega) = \frac{e^2}{\varepsilon_0 m} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

Physics behind γ

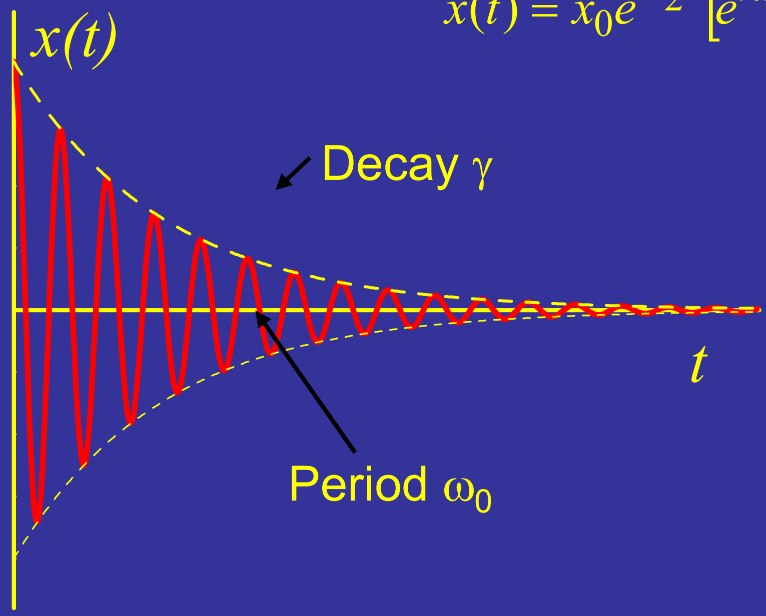
No electric field: $$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0; \qquad \omega_0^2 = \frac{k}{m}$$



Solution: $x(t) = x_0 e^{-i\omega t}$

$$-\omega^2 - i\gamma\omega + \omega_0^2 = 0; \qquad \omega = \frac{i\gamma \pm \sqrt{4\omega_0^2 - \gamma^2}}{2} \overset{\gamma << \omega_0}{\cong} \frac{i\gamma}{2} \pm \omega_0$$

$$x(t) = x_0 e^{-\frac{\gamma}{2}t}\left[e^{i\omega_0 t} + e^{-i\omega_0 t}\right] = 2x_0 e^{-\frac{\gamma}{2}t}\cos\omega_0 t$$



Free oscillations of the dipole decay with a characteristic time γ

Mechanical analog – friction
But: what is friction for a dipole?

Radiative lifetime

Total energy: $U(t) = \frac{kx^2(t)}{2} + \frac{mv^2(t)}{2} = kx_0^2 e^{-\gamma t}\cos^2\omega_0 t + mx_0^2 e^{-\gamma t}\,\omega_0^2 \sin^2\omega_0 t \overset{\omega_0^2 = \frac{k}{m}}{=} mx_0^2\omega_0^2 e^{-\gamma t}$

$U(t) = U_0 e^{-\gamma t}$, where $U_0 = mx_0^2\omega_0^2$

Power: $P = \frac{dU(t)}{dt} = -\gamma U_0 e^{-\gamma t} \cong -\gamma U_0 = -\gamma m x_0^2 \omega_0^2$

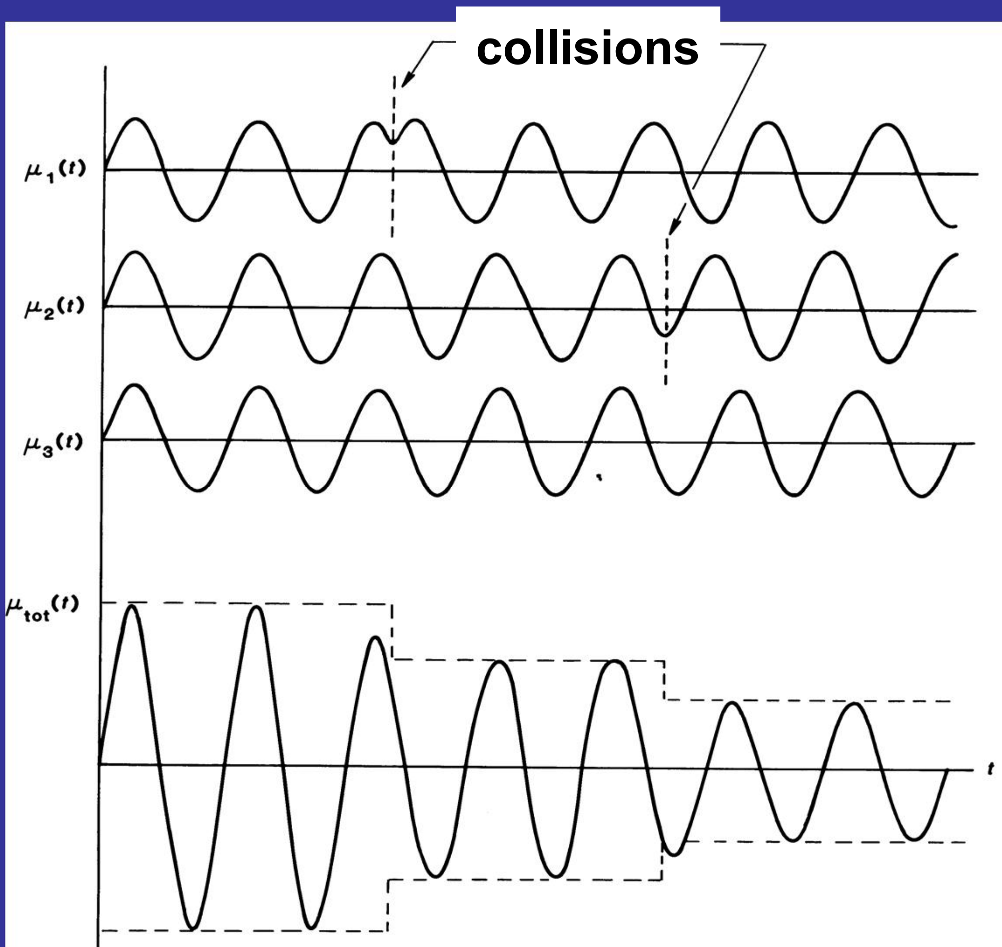
Power radiated by a dipole: $P = \frac{1}{4\pi\varepsilon_0}\frac{p_0^2\omega_0^4}{3c^3} = \frac{1}{4\pi\varepsilon_0}\frac{x_0^2 e^2\omega_0^4}{3c^3}$

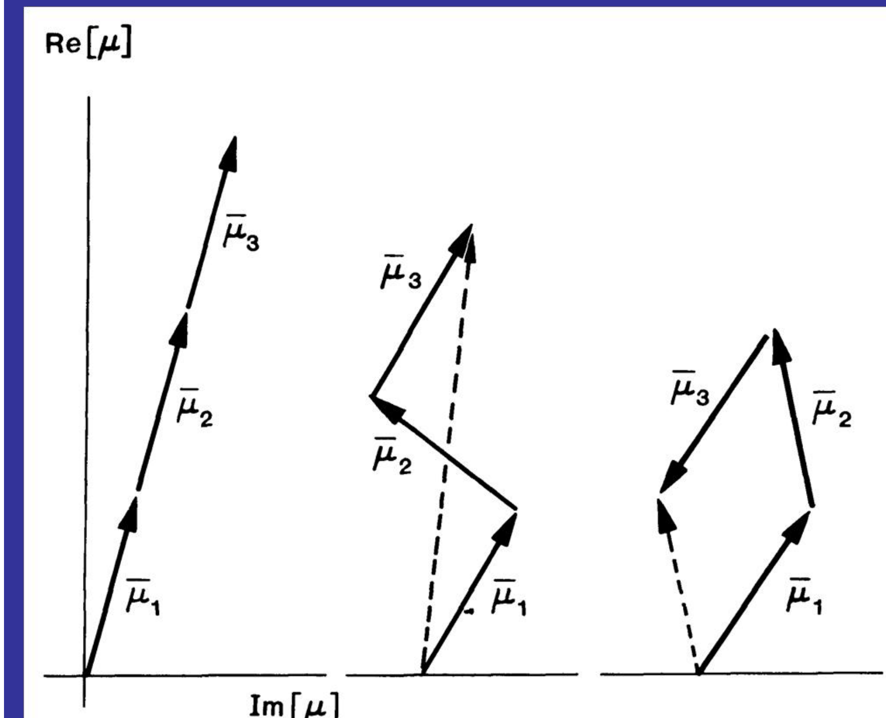
$$\gamma m x_0^2\omega_0^2 = \frac{1}{4\pi\varepsilon_0}\frac{x_0^2 e^2 \omega_0^4}{3c^3}; \qquad \gamma = \frac{e^2\omega_0^2}{12\pi\varepsilon_0 mc^3}$$

Radiative lifetime: $\tau \equiv \frac{1}{\gamma} = \frac{12\pi\varepsilon_0 mc^3}{e^2\omega_0^2}$

If $\omega_0 \cong 4\cdot 10^{15}\,s^{-1}$, $\gamma \cong 10^8\,s^{-1}$ and $\tau \cong 10^{-8}\,s = 10\,ns$

Dephasing collisions





Dephasing time T_2

Initial polarization: $P(t=0) = N_0 \mu_0$

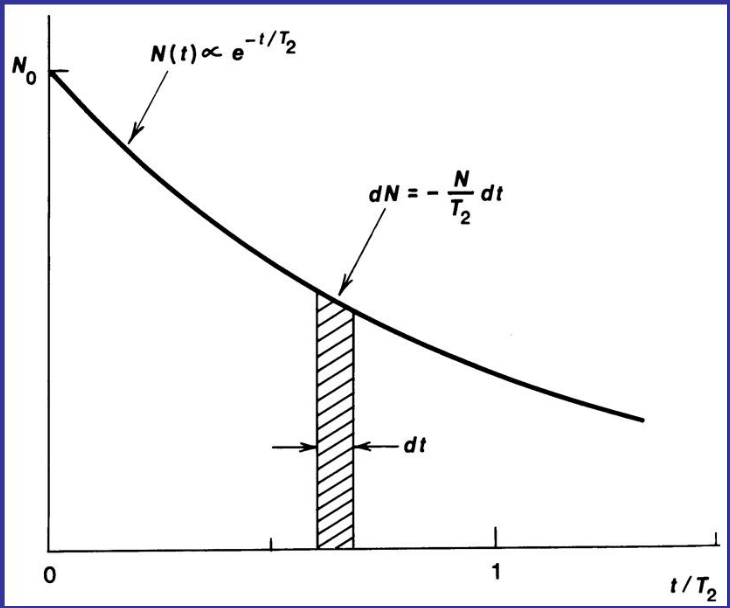
Let's assume that $\gamma = 0$: $\mu(t) = \mu_0 \cos \omega t$

N(t) dipoles have not suffered a single collision by time t

$$P(t) = N(t)\mu(t) = N(t)\mu_0 \cos \omega t$$

Collision rate $1/T_2$ per atom per second

$$dN(t) = -\frac{N(t)}{T_2} dt \quad ; \quad N(t) = N_0 e^{-\frac{t}{T_2}}$$



$$P(t) = N_0 e^{-\frac{t}{T_2}} \mu_0 \cos \omega t = N_0 e^{-\frac{t}{T_2}} \times \text{old result} \overset{\gamma \neq 0}{=} N_0 e^{-\frac{t}{T_2}} \mu_0 e^{-\frac{\gamma t}{2}} \cos \omega t$$

$$\frac{\gamma}{2} \Rightarrow \frac{\gamma}{2} + \frac{1}{T_2}$$

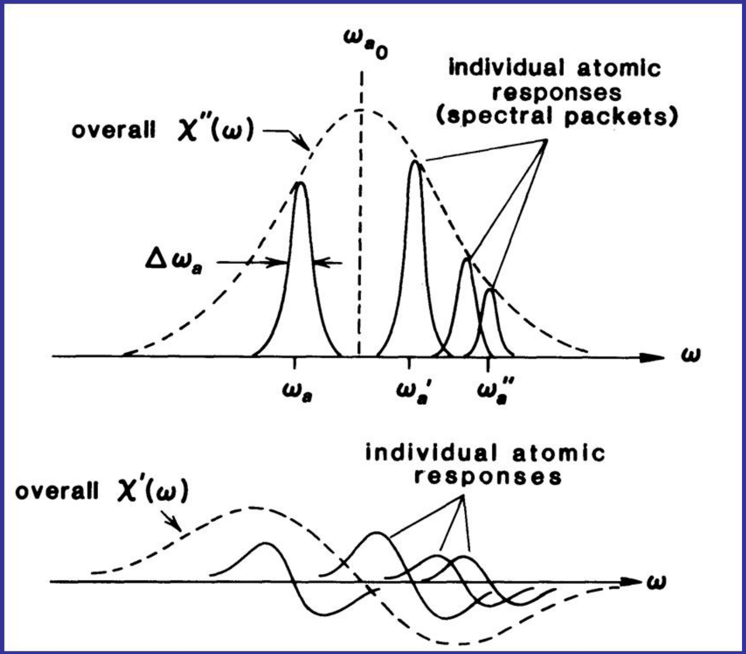
T_2 - dephasing time
(phase but not energy relaxation)

Processes leading to T_2
- Collisions
- Thermal vibrations
- Dipolar coupling

Inhomogeneous Lineshape

Certain distribution of oscillator frequencies ω_a in $\chi(\omega,\omega_a) \propto \dfrac{1}{(\omega_a - \omega - i\gamma/2)}$

Why?
- -Doppler broadening
- -Different surrounding



Let's assume Gaussian distribution:

$$g(\omega_a) = \sqrt{\frac{1}{\pi\,\Delta\omega^2}}\; e^{-\frac{(\omega_0-\omega_a)^2}{\Delta\omega^2}}$$

ω_0 - center frequency

$\Delta\omega$ - distribution width

Susceptibility: $\tilde{\chi}(\omega) \propto \int g(\omega_a)\chi(\omega,\omega_a)\,d\omega_a = \int e^{-\frac{(\omega_0-\omega_a)^2}{\Delta\omega^2}} \dfrac{1}{(\omega_a - \omega - i\gamma/2)} d\omega_a$

Strongly homogeneous limit: $\gamma >> \Delta\omega$; $\omega_0 \cong \omega_a$ $\Rightarrow$ old result

Strongly inhomogeneous limit: $\gamma << \Delta\omega$

Inhomogeneous Lineshape

Susceptibility: $$\tilde{\chi}(\omega) \propto \int g(\omega_a)\chi(\omega,\omega_a)\,d\omega_a = \int e^{-\frac{(\omega_0-\omega_a)^2}{\Delta\omega^2}} \frac{1}{(\omega_a-\omega+i\gamma/2)}\,d\omega_a$$

Strongly inhomogeneous limit: $\gamma << \Delta\omega$

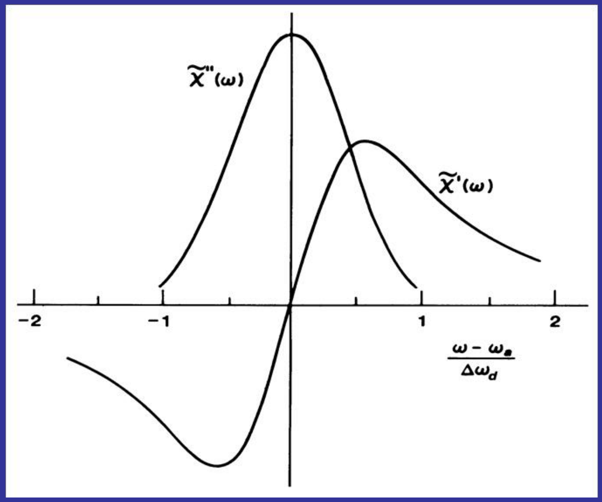
Im part: $$\tilde{\chi}''(\omega) \propto \int e^{-\frac{(\omega_0-\omega_a)^2}{\Delta\omega^2}} \frac{\gamma/2}{(\omega_a-\omega)^2+(\gamma/2)^2}\,d\omega_a \overset{\omega_a \cong \omega}{\propto} e^{-\frac{(\omega-\omega_0)^2}{\Delta\omega^2}}$$

Re part: $$\tilde{\chi}'(\omega) \propto \int e^{-\frac{(\omega_0-\omega_a)^2}{\Delta\omega^2}} \frac{(\omega_0-\omega)}{(\omega_0-\omega)^2+(\gamma/2)^2}\,d\omega_a \overset{\omega_a \cong \omega}{\propto} \;?$$ - numerical calculations

For large detunings $\delta\omega$ from ω_0 :

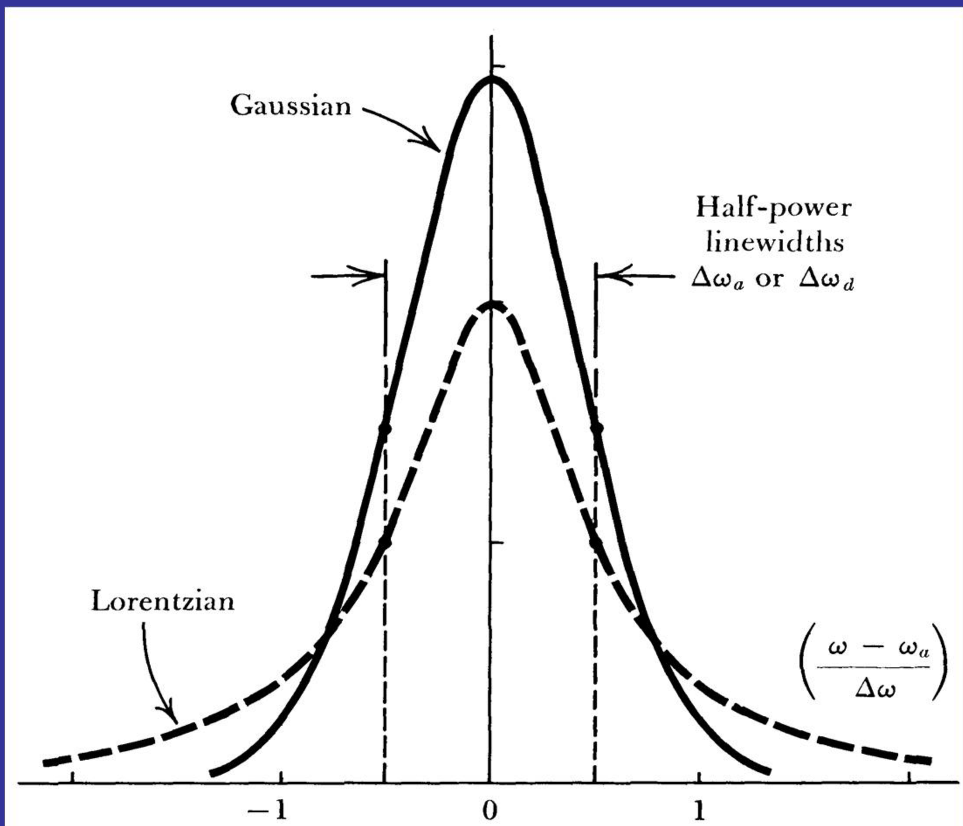
Absorption (Im part): $\propto e^{-\frac{\delta\omega^2}{\Delta\omega^2}}$

Refraction (Re part): $\propto \frac{1}{\delta\omega}$

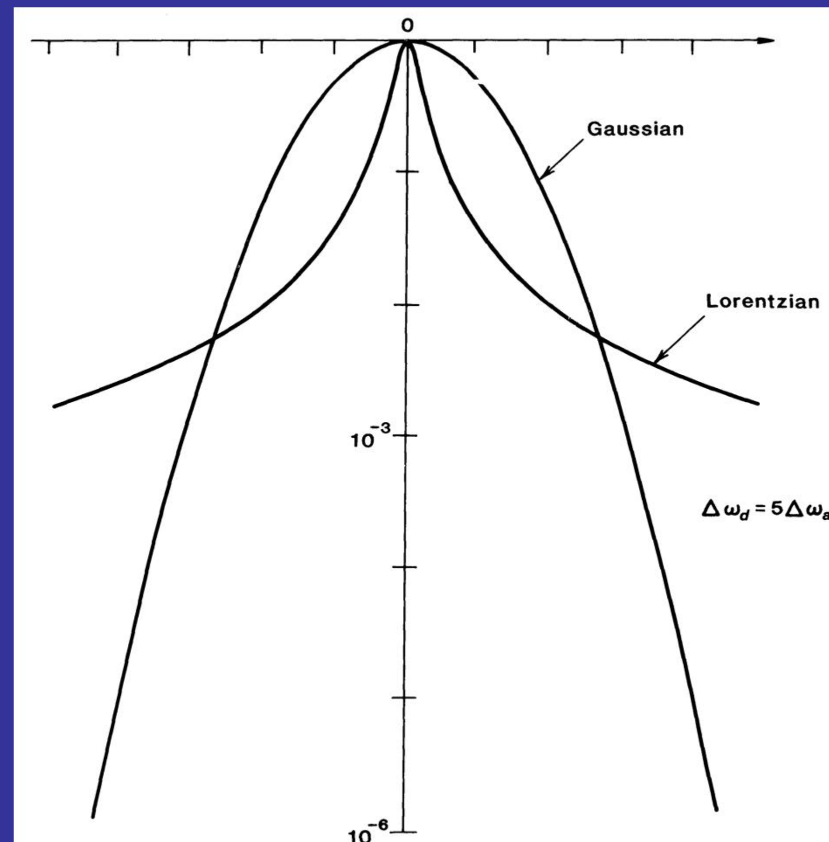


Homogeneous vs Inhomogeneous Lineshapes

Linear scale



Logarithmic scale

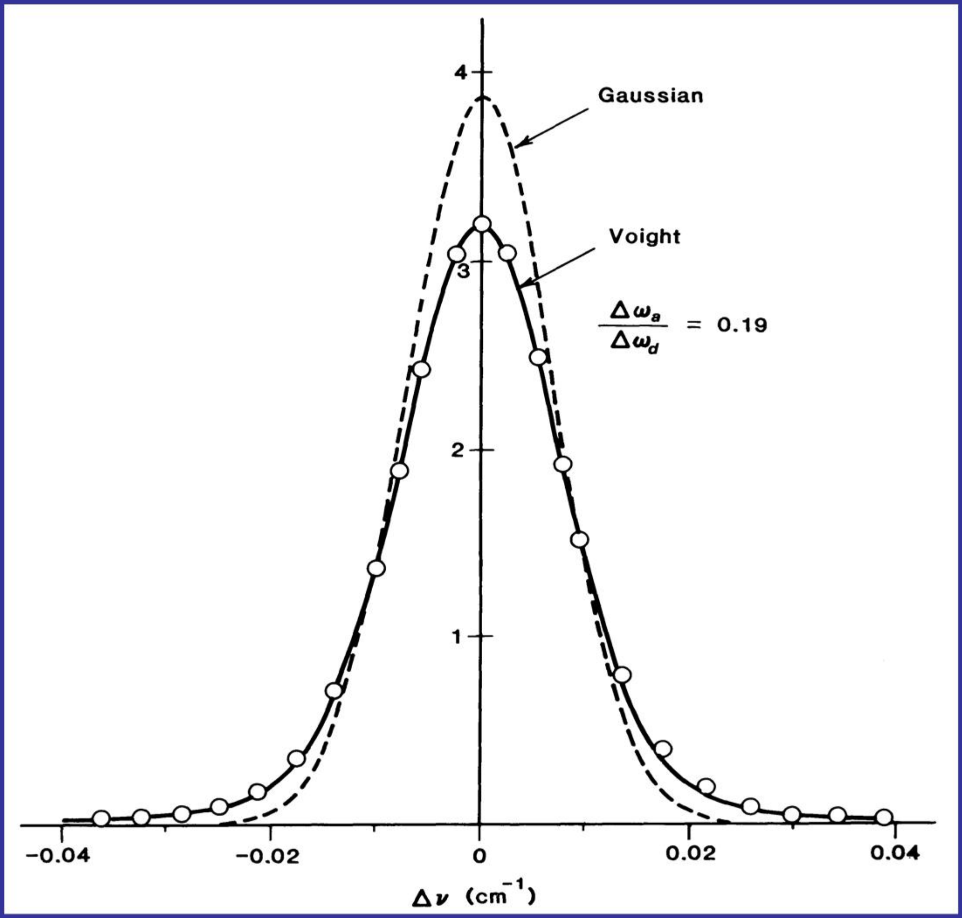


Homogeneous vs Inhomogeneous Lineshapes

Voight profile :

Numerical convolution:

$$\tilde{\chi}''(\omega) \propto \int e^{-\frac{(\omega_0-\omega_a)^2}{\Delta\omega^2}} \frac{\gamma/2}{(\omega_a-\omega)^2+(\gamma/2)^2}\, d\omega_a$$



Kramers kronig

Kramers-Kronig Relations

$$\varepsilon(\omega) = n^2(\omega) \cong 1 + \frac{Ne^2}{2\varepsilon_0 V m \omega_0} \frac{(\omega_0 - \omega) - i\gamma/2}{(\omega_0 - \omega)^2 + (\gamma/2)^2}$$

Relation between real and imaginary parts of $\varepsilon(\omega)$

$$\varepsilon'(\omega) = 1 + \frac{Ne^2}{2\varepsilon_0 V m \omega_0} \frac{(\omega_0 - \omega)}{(\omega_0 - \omega)^2 + (\gamma/2)^2}; \qquad \varepsilon''(\omega) = \frac{Ne^2}{2\varepsilon_0 V m \omega_0} \frac{\gamma/2}{(\omega_0 - \omega)^2 + (\gamma/2)^2}$$

$$\varepsilon'(\omega) = 1 + \varepsilon''(\omega) \frac{2(\omega_0 - \omega)}{\gamma}$$

True in general:

$$\varepsilon'(\omega) = 1 + \frac{2}{\pi} \int_0^{+\infty} \frac{\omega' \varepsilon''(\omega')}{\omega'^2 - \omega^2} d\omega'$$

$$\varepsilon''(\omega) = -\frac{2\omega}{\pi} \int_0^{+\infty} \frac{\varepsilon'(\omega') - 1}{\omega'^2 - \omega^2} d\omega'$$

Causal connection between the polarization and electric field

Derivation of absorption from refraction and *vice versa*

Kramers Kronig relations

Based on causality.
Only works if one knows the *complete* spectrum of one function!

Real and imaginary parts of susceptibility

$$\chi'(\omega) = \frac{2}{\pi}\int_0^\infty \frac{\omega'\chi''(\omega')}{\omega'^2-\omega^2}\,d\omega'$$

Real part of the refractive index and the absorption coefficient

$$n'(\omega) = 1 + \frac{c}{\pi}\int_0^\infty \frac{\alpha(\omega')}{\omega'^2-\omega^2}\,d\omega'$$

Phase shift upon reflection and the reflectivity

$$\varphi(\omega) = \frac{\omega}{\pi}\int_0^\infty \frac{\ln\left(R(\omega')\right)-\ln\left(R(\omega)\right)}{\omega'^2-\omega^2}\,d\omega'$$

- Derive the wave equation
- 1.8; 1.12; 1.19

- Derive the response function of a Lorentz oscillator
- 2.3; 2.6;

- 7.1, 7.6, 7.7;