Notes on 2 DEG in a magnetic field

Magnetic field along z-direction, two dimensional electron gas (2DEG) in the xy-plane.

Canonical momentum of a particle in a field: $\vec{p} = m\vec{v} + e\vec{A}$ (see lagrangian treatment in classical electrodynamics). In quantum mechanics this reads: $\vec{p} = -i\hbar\vec{\nabla} + e\vec{A}$.

The QM Hamilton operator then becomes

$$\hat{H} = \frac{1}{2m}\left(\hat{p} + e\vec{A}\right)^2 - e\phi(\vec{x}, t)$$

with ϕ the electric potential (which we do not need here).

Vector field $\vec{A}$ in our case ($\vec{B} = (0, 0, B_z)$), using the landau gauge, is given by $\vec{A} = (-B_z y, 0, 0)$, yielding ($B = B_z$):

$$\hat{H} = \frac{1}{2m}\left[(\hat{p}_x - eBy)^2 + \hat{p}_y^2 + \hat{p}_z^2\right]$$

It is obvious that for this Hamilton operator, since $[H, \hat{p}_x] = [H, \hat{p_z}] = 0$, p_x and p_z are conserved quantities. The wave function can thus be written as

$$\psi\left(x\right) = e^{i(p_x x + p_z z)/\hbar}\phi(y),$$

with

$$\left[\frac{\hat{p}_y^2}{2m} + \frac{1}{2}m\omega_c^2\left(y - y_0\right)^2\right]\phi(y) = \left(E - \frac{p_z^2}{2m}\right)\phi(x), \tag{1}$$

where $y_0 = \frac{p_x}{eB}$ and the classical cyclotron frequency $\omega_c = \frac{eB}{m}$.

Equation 1 obviously has the form of a harmonic oscillator in y with energy

$$E = \left(n + \frac{1}{2}\right)\hbar\omega_c + \frac{p_z^2}{2m} = \left(n + \frac{1}{2}\right)\hbar\omega_c. \tag{2}$$

The latter equality comes from the fact that we are considering a 2DEG restricted to the xy plane.

Density of states (DOS)

Without a field we have for the 2DEG a density of states given by

$$D(E) = \frac{\partial N}{\partial k}\frac{\partial k}{\partial E} = \left(\frac{k}{2\pi}\right)\left(\frac{m}{\hbar^2 k}\right) = \frac{m}{2\pi\hbar^2}. \tag{3}$$

In a magnetic field this DOS will change due to the quantization of the orbits and, hence, the energy. In fact it will split up from the constant value for $B = 0$ to discrete peaks at values $E_n = (n + 1/2)\hbar\omega_c$ The magnetic field will not alter the total number of states, and there is no reason why one peak should get more peaks than the other. In fact all states in the range $\pm\hbar\omega_c/2$ around E_n will contribute to the number of states for level n:

$$N_n = \hbar\omega_c \cdot \frac{m}{2\pi\hbar^2} = \hbar\frac{eB}{m}\frac{m}{2\pi\hbar} = \frac{eB}{h}.$$

The number of states in a level depends linearly on the magnetic field B. We can now define a so-called filling factor:

$$\nu = N_{tot}/N_n = \frac{h}{e}N_{tot}\frac{1}{B}$$

If we compare this to the classical result for the Hall effect we found before: $\sigma_{xy} = \frac{N_{tot}e}{B}$ we see that we can also write this as

$$\sigma_{xy} = \frac{\nu\frac{eB}{h}e}{B} = \nu\frac{e^2}{h} = \nu \cdot [25833.81\ \Omega]^{-1}$$

As a function of field, the number of filled Landau levels changes (as $1/B$).

$$B_\nu = \frac{h}{e}\frac{N_{tot}}{\nu}$$

The field difference between integer numbers of filled levels is then:

$$\Delta\left(\frac{1}{B}\right) = \frac{1}{B_{\nu+1}} - \frac{1}{B_\nu} = \frac{1}{N_{tot}}\frac{e}{h}.$$