London equations

repeat: Drude (~ 1900)

1. e^- : gas without coulomb interactions (screening).
2. independent e^- approximation → calculate response single e^-
3. e^- undergo collisions randomizing $\vec{t}$
4. thermal equil. with lattice (i.e. total E conserved).
5. local response.

E.O.M. : $m\dot{v} = F_{EM} + F_{scatt}$.

$F_{EM} = q(\bar{E} + (\bar{v} \times \vec{B}))$ — $\bar{E}$ ↓ ohm, $(\bar{v} \times \vec{B})$ ↓ Hall.

$F_{scatt} = -\frac{m}{\tau}\bar{v}$ — τ: transport scattering time

no B-field:

$$m\dot{v} + \frac{m}{\tau}v = qE.$$

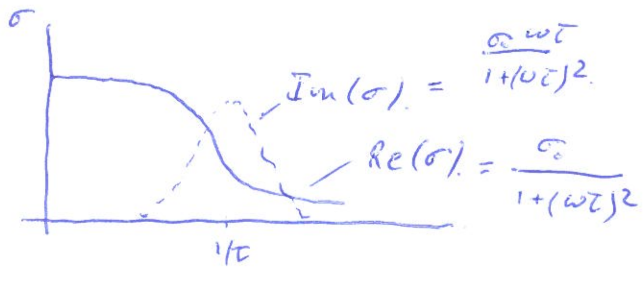
for time varying fields : $\bar{E} = E_0 e^{i\omega t}$, $v = v_0 e^{i\omega t}$.

$$\Rightarrow i\omega m v_0 + \frac{m}{\tau}v_0 = qE_0.$$

$$v_0 = \left(\frac{q\tau}{m}\right) \cdot \frac{1}{1+i\omega\tau} E_0.$$

Current density : $\vec{J}_0 = nq\vec{v}_0 = \left(\frac{nq^2\tau}{m}\right)\frac{1}{1+i\omega\tau}E_0$ $\qquad \sigma_0 = \frac{nq^2\tau}{m}$

$$\sigma = \sigma_0 \frac{1}{1+i\omega\tau}.$$



$\tau \sim 10^{-14}$ sec.

$\omega\tau \ll 1$ even at a THz → $\omega <$ THz $\sigma = \sigma_0$

perfect conductor.

$\omega\tau >> 1 \qquad (\tau \to \infty)$

$$\sigma(\omega) = \frac{\sigma_0}{1+i\omega\tau} = \frac{1}{i\omega\left(\frac{m}{nq^2}\right)} \qquad \text{perfect "inductor"}$$

perfectly conducting regime $\sigma = \sigma_0 \to \infty$

Cooper pair: $\left.\begin{array}{l} m^* = 2m_e \\ q^* = 2q_e \\ n^* = n/2 \end{array}\right\} \quad \sigma = \dfrac{1}{i\omega\left(\frac{m^*}{n^* q^{*2}}\right)} = \dfrac{1}{i\omega\Lambda} \qquad \Lambda = \dfrac{m^*}{n^* q^{*2}}$

$$J(\omega) = \sigma \cdot E = \frac{1}{i\omega\Lambda} E(\omega) \Rightarrow \boxed{E = \frac{\partial}{\partial t}(\Lambda J)}$$ 1st London equation.

for static E fields.

Ampère law (4th Maxwell eq.): $\nabla \times H = J$

$$\nabla \times \nabla \times H = \nabla \times J$$

$$\frac{\partial}{\partial t}\Big(\underbrace{\nabla(\nabla\cdot H)}_{=0} - \nabla^2 H = \nabla \times J\Big)$$

(gauss law, magnetism, 2nd Maxwell).

using $E = \frac{\partial}{\partial t}(\Lambda J)$: $\cancel{\nabla \times \frac{E}{\Lambda}} - \nabla^2 \frac{\partial H}{\partial t} = \nabla \times \frac{E}{\Lambda}$

since $\nabla \times E = -\frac{\partial}{\partial t}\mu_0 H$.

(Faraday's induction law, 3rd Maxwell)

one has

$$\boxed{\left(\frac{\mu_0}{\Lambda} - \nabla^2\right)\frac{\partial}{\partial t} H = 0}$$ for a perfect conductor.

$$\left(\frac{\mu_0}{\Lambda} - \nabla^2 \right) \frac{\partial}{\partial t} H = 0.$$

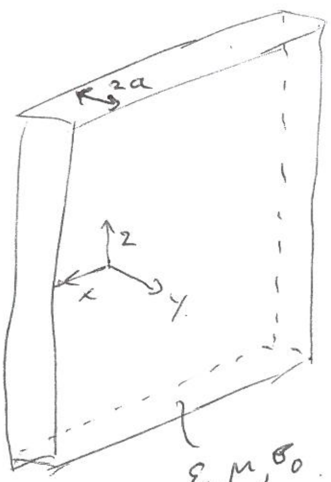
or $\left(\frac{1}{\lambda^2} - \nabla^2 \right) \frac{\partial}{\partial t} H = 0.$ $\quad \lambda = \sqrt{\frac{\Lambda}{\mu_0}}$: penetration depth.

$H(z) = H_0 e^{-z/\lambda}$ indep. of ω $\quad$ frequency independent

$$\Lambda = \frac{m}{nq^2}$$

	λ (nm)
Sn	34
Al	16
pb	37
Cd	110
Nb	39

perfectly conducting infinite slab, thickness $2a$.



$$\bar{H}_{\text{applied}}(\bar{r}, t) = H_0 e^{i\omega t} \vec{e}_z$$

take $\bar{H}(\bar{r}, t) = \bar{H}(y) e^{i\omega t} \bar{e}_z$

then $i\omega \left(\frac{1}{\lambda^2} - \frac{d^2}{dy^2} \right) H(y) = 0.$

or $H(y) = C \cdot \cosh(y/\lambda)$

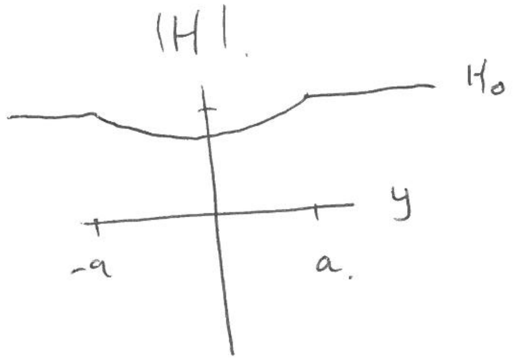
boundary conditions : $H(y=a) = H(y=-a) = H_0.$

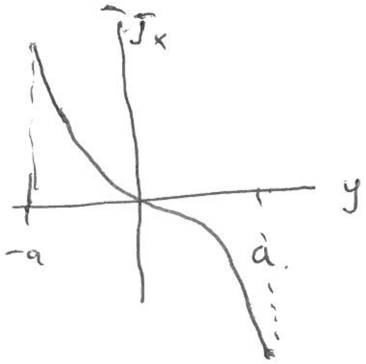
so that $H(\bar{r}, t) = H_0 \frac{\cosh(y/a)}{\cosh(a/\lambda)} e^{i\omega t} \bar{e}_z$

using amperes law ($\nabla \times H = J$; assuming $\frac{\partial E}{\partial t} = 0$).

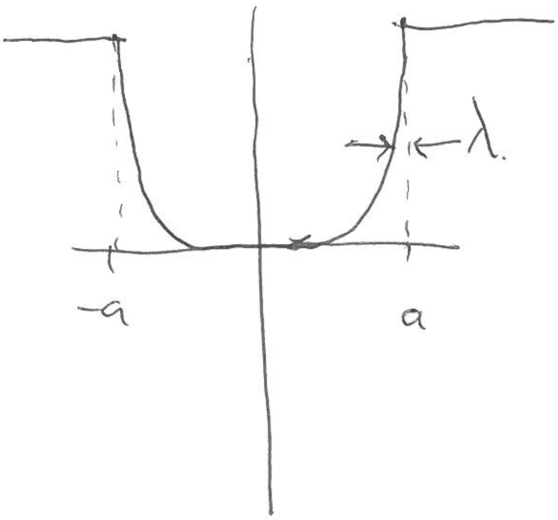
we get $\bar{J} = \frac{1}{\lambda} H_0 \frac{\sinh(y/\lambda)}{\cosh(y/a)} e^{i\omega t} \bar{e}_x$

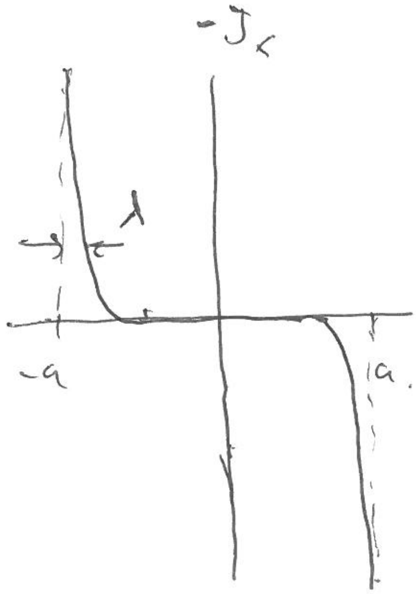
Thin film limit : $a \ll \lambda$.





Bulk limit





2^{nd}. London eq. follows from.

faraday's induction law $\nabla \times E = -\frac{\partial}{\partial t}\mu_0 H$.

and the 1^{st} London eq. $E = \frac{\partial}{\partial t}(\Lambda J)$.

$$\nabla \times E = -\frac{\partial}{\partial t}\mu_0 H.$$

$$\nabla \times \left(\frac{\partial}{\partial t}\Lambda J\right) = -\frac{\partial}{\partial t}\mu_0 H.$$

assuming we can interchange space/time ~~rate~~ derivs,

$$\boxed{\nabla \times \Lambda J = -B}$$ 2^{nd} London equation.

London eq^s from s.c. wfunct.

s.c. wfunct: $\Psi = \sqrt{n_s}\, e^{i\varphi}$.

in field: canonical momentum $p = \frac{\hbar}{i}\bar{\nabla} - q\bar{A}$

current wfunct.: $\bar{J} = q \cdot \bar{v} = \frac{q}{2m}\left\{ \psi^*\left(\frac{\hbar}{i}\bar{\nabla} - q\bar{A}\right)\psi + c.c.\right\}$.

$= \frac{q}{m} n_s \cdot \left\{ \hbar \nabla\varphi - q\bar{A} \right\}$.

$\Delta\varphi$ <u>and</u> $\bar{A}$ drive current (cmp. Josephson effect)
(also e^{ikr} $\nabla\varphi = \bar{k}$)

when no phase jumps & singly connected geometry.
choose gauge such that φ = constant (rigid gauge).

then. $\bar{J} = -\frac{q^2}{m} n_s \bar{A}$

assumption London: $\boxed{\bar{J} = -\frac{1}{\lambda^2\mu_0}\bar{A}}$ $\Rightarrow \lambda = \sqrt{\frac{m}{\mu_0 q^2 n_s}} = \sqrt{\frac{\varepsilon_0 m c^2}{q^2 n_s}}$.

1st London eq.: $E = -\nabla V - \frac{\partial A}{\partial t}$.

for $\nabla V = 0$:

$$\boxed{\frac{\partial \bar{J}}{\partial t} = \frac{1}{\lambda^2\mu_0}\frac{\partial \bar{A}}{\partial t} = \frac{1}{\lambda^2\mu_0} E}$$ 1st

2^n London eq. $B = \nabla \times A$

$\nabla \times J = -\frac{1}{\lambda^2\mu_0}\nabla \times A$

or $\boxed{\nabla \times \bar{J} = -\frac{1}{\lambda^2\mu_0} B}$ 2nd.