as before sum over k.

$$\sum_k g_k = \sum_k \frac{VC}{2\varepsilon_k - \lambda} \quad \Rightarrow \quad 1 = \sum_k \frac{V}{2\varepsilon_k - \lambda} \qquad (a)$$

similar eq^{n} as before, is λ then also E?
from bottom page (6), taking compl. conjugate.

$$(2\varepsilon_k - \lambda^*) g_k^* = V \sum_k g_k^*$$

multiply by g_k & sum over k.

$$\sum_k (2\varepsilon_k - \lambda^*) g_k^* g_k = V \sum_k \sum_{k'} g_{k'}^* g_k$$

$$\sum_k |g_k|^2 \cdot 2\varepsilon_k - \lambda \underbrace{\sum_k |g_k|^2}_{=1} = V \sum_{k,k'} g_{k'}^* g_k$$

$$\Rightarrow \lambda = \sum_k 2\varepsilon_k |g_k|^2 - V \sum_k \sum_{k'} g_{k'}^* g_k.$$

top page (6) : $E = 2\sum_k \varepsilon_k |g_k|^2 - V \sum_{k,q} g_{k+q}^* g_k$.

relabel $k+q$ to k'.

so $\lambda = E$.

and (a) becomes

$$1 = \sum_k \frac{V}{2E - \lambda}$$ same eq. as in wavefunction approach

$\Rightarrow$ bound state with

$$E = 2\varepsilon_F - \frac{2\hbar\omega_0\, e^{-\frac{2}{V_0 N(E_F)}}}{1 - e^{-2/V_0 N(E_F)}}$$ (see page 3)

Bogoliubov approach to BCS.

- BCS 1957 paper took variational approach. here we take 2nd quant. diagonalization approach.

- BCS hamilton operator:

$$H_{BCS} = \sum_{k,\sigma} \varepsilon_k c^+_{k\sigma} c_{k\sigma} - \frac{1}{2} \sum_{\substack{kk'\sigma \\ \sigma'}} V_{kk'} c^+_{k'\sigma'} c^+_{-k'\sigma} c_{-k\sigma} c_{k\sigma'}$$

$V_{kk'}$ contains e-p interaction and coulomb repulsion, both independent of σ

$$V_{kk'} = -2W_{k,k',k'-k} - U_{kk'}$$

screened coulomb.

$$U_{k,k'} = U(k-k') = U(q) = \frac{4\pi e}{q^2 + k_s^2}$$

Thomas fermi screening $\varepsilon = 1 + \frac{k_s^2}{q^2}$.

$U_{kk'}$ always positive: so that it always has + contribution to H_{BCS} $\Rightarrow$ can not lead to pairing e-p term $$W_{k,k',q} = \frac{|M_q|^2 \hbar\omega_q}{(\varepsilon_k - \varepsilon_{k'})^2 - (\hbar\omega_q)^2}$$ (from 2nd order perturb.)

(from Fröhlich interaction.
$$M_q = i\sqrt{\frac{N\hbar}{2M_{ion}\omega_q}}\,|q|\,V_q$$
V_q ← fourier transform ionic potential.

$N = \#$ ions

for singlet pairing we can sum over σ, σ' (i.e. $\sigma' = -\sigma$; 2 variants $\sigma = +$ or $\sigma = -$)

e.g. $$\sum_\sigma c^+_{k'\sigma'} c^+_{-k'\sigma} c_{-k\sigma} c_{k\sigma'} =$$
$$c^+_{k'\uparrow} c^+_{-k'\downarrow} c_{-k\downarrow} c_{k\uparrow} +$$
$$c^+_{k'\downarrow} c^+_{-k'\uparrow} c_{-k\uparrow} c_{k\downarrow}$$
$$= c^+_{k'\uparrow} c^+_{-k'\downarrow} c_{-k\downarrow} c_{k\uparrow} + c^+_{-k'\uparrow} c^+_{k'\downarrow} c_{k\downarrow} c_{-k\uparrow}.$$

same $\Rightarrow \sum_\sigma \rightarrow$ factor 2.
and if $-k$ then $\downarrow$

$$\Rightarrow \boxed{H_{BCS} = \sum_k \varepsilon_k \big(\underbrace{c^+_k c_k}_{\uparrow} + \underbrace{c^+_{-k} c_{-k}}_{\downarrow}\big) - \sum_{kk'} V_{kk'} c^+_{k'} c^+_{-k'} c_{-k} c_k}$$

BCS Hamilton operator.

difficult term since off diagonal

can not take perturbative approach G.S. nature changes! pairs!

Bogoliubov - Valatin found transformation that simplifies ~~pa~~ H_{BCS} and makes it tractable.

starts with definition of new operators. (fermionic)

$$\gamma_k = u_k c_k - v_k c_{-k}^+$$ "single particle" operators

$$\gamma_{-k} = u_k c_{-k} + v_k c_k^+$$

for now: u_k, v_k real so that.

$$\gamma_k^+ = u_k c_k^+ - v_k c_{-k}$$

$$\gamma_{-k}^+ = u_k c_{-k}^+ + v_k c_k$$

to make these fermionic operators
i.e. $\{\gamma_k, \gamma_{k'}\} = \{\gamma_k, \gamma_{-k'}\} = \{\gamma_k^+, \gamma_{-k}\} = 0$

and $\{\gamma_k^+, \gamma_{k'}\} = \{\gamma_{-k}^+, \gamma_{-k'}\} = \delta_{kk'}$

one needs $u_k^2 + v_k^2 = 1$.

'old' operators in terms of new ones:

$$c_k = u_k \gamma_k + v_k \gamma_{-k}^+ \qquad c_k^+ = u_k \gamma_k^+ + v_k \gamma_{-k}.$$

$$c_{-k} = u_k \gamma_{-k} - v_k \gamma_k^+ \qquad c_{-k}^+ = u_k \gamma_{-k}^+ - v_k \gamma_k.$$

now we can rewrite H_{BCS} in terms of new operators first kin. E term $\sum_k \varepsilon_k (c_k^+ c_k + c_{-k}^+ c_{-k})$.

$$H_{kin} = \sum_k \varepsilon_k \Big[\{u_k \gamma_k^+ + v_k \gamma_{-k}\}\{u_k \gamma_k + v_k \gamma_{-k}^+\} + \{u_k \gamma_{-k}^+ - v_k \gamma_k\}\{u_k \gamma_{-k} - v_k \gamma_k^+\} \Big]$$

$$= \sum_k \varepsilon_k \Big[u_k^2 \gamma_k^+ \gamma_k + v_k u_k \gamma_{-k} \gamma_k + u_k v_k \gamma_k^+ \gamma_{-k}^+ + v_k^2 \gamma_{-k} \gamma_{-k}^+ + u_k^2 \gamma_{-k}^+ \gamma_{-k} - u_k v_k \gamma_k \gamma_{-k} - u_k v_k \gamma_{-k}^+ \gamma_k^+ + v_k^2 \gamma_k \gamma_k^+ \Big]$$

diagonal terms:

$$u_k^2 \gamma_k^+ \gamma_k + v_k^2 \gamma_{-k} \gamma_{-k}^+ + u_k^2 \gamma_{-k}^+ \gamma_{-k} + v_k^2 \gamma_k \gamma_k^+.$$

since $\{\gamma_k^+, \gamma_{k'}\} = \{\gamma_{-k}^+, \gamma_{-k'}\} = \delta_{kk'}$.

one has $\gamma_k^+ \gamma_k + \gamma_k \gamma_k^+ = 1$.

and diag. terms become.

$$2v_k^2 + (u_k^2 - v_k^2)\big(\underbrace{\gamma_k^+ \gamma_k}_{\text{number operator } m_k} + \gamma_{-k}^+ \gamma_{-k}\big) =$$

$$= 2v_k^2 + (u_k^2 - v_k^2)(m_k + m_{-k})$$

so kin. E term becomes

$$H_{BCS}^{kin} = \sum_k \varepsilon_k \Big[2v_k^2 + (u_k^2 - v_k^2)(m_k + m_{-k}) + 2u_k v_k (\gamma_k^+ \gamma_{-k}^+ + \gamma_{-k}\gamma_k) \Big]$$

i.e. constant + diagonal term in number operators
+ <u>off diagonal term.</u>

trick is to make this zero => diagonal
can be done with interacting part.

interacting part:

$$H_{BCS}^{V} = -\sum_{kk'} V_{kk'} c_{k'}^+ c_{-k'}^+ c_{-k} c_k = -\sum_{kk'} V_{kk'} (u_{k'} \gamma_{k'}^+ + v_{k'} \gamma_{-k'}) \times (u_{k'} \gamma_{-k'}^+ - v_{k'} \gamma_{k'}) \times (u_k \gamma_{-k} - v_k \gamma_k^+) \times (u_k \gamma_k + v_k \gamma_{-k}^+)$$

$$= -\sum_{kk'} V_{kk'} \Big[u_{k'} v_{k'} u_k v_k (1 - m_{k'} - m_{-k'})(1 - m_k - m_{-k}) + u_{k'} v_{k'} (1 - m_{k'} - m_{-k'})(u_k^2 - v_k^2)(\gamma_{-k}\gamma_k + \gamma_k^+ \gamma_{-k}^+)$$

+ fourth order off diagonal terms.

if ~~the~~ non-diagonal terms H^{kin} & H^V cancel then we are done

① is ground state: $m_k = m_{-k} = 0$. no excited states.
(one can check this later).

using $m_k = m_{-k} = 0$ the off-diagonal parts become:

$$\sum_k 2\varepsilon_k u_k v_k \left(\gamma_k^+ \gamma_{-k}^+ + \gamma_{-k}\gamma_k\right) - \sum_{kk'} V_{kk'} u_{k'} v_{k'} \times (u_k^2 - v_k^2)(\gamma_k^+ \gamma_{-k}^+ + \gamma_{-k}\gamma_k) + 4^{th}\text{ order term} \equiv 0$$

② we'll approx. 4^{th} order $\rightarrow 0$ (can also be checked later-on)

then $$2\varepsilon_k u_k v_k - (u_k^2 - v_k^2)\sum_{k'} V_{kk'} u_{k'} v_{k'} = 0.$$

since $u_k^2 + v_k^2 = 1$ we can eliminate one of them.

using $u_k = \left(\frac{1}{2} - x_k\right)^{1/2}$ $v_k = \left(\frac{1}{2} + x_k\right)^{1/2}$

one gets.

$$2\varepsilon_k \left(\frac{1}{4} - x_k^2\right)^{1/2} + 2x_k \underbrace{\sum_{k'} V_{kk'}\left(\frac{1}{4} - x_{k'}^2\right)^{1/2}}_{\text{def. as } \Delta_k} = 0.$$

$$\Rightarrow x_k = \pm \frac{\varepsilon_k}{2(\varepsilon_k^2 + \Delta_k^2)^{1/2}}.$$

plugging this back into def. of Δ_k gives

$$\Delta_k = \frac{1}{2}\sum_{k'} V_{kk'} \cdot \frac{\Delta_{k'}}{(\varepsilon_{k'}^2 + \Delta_{k'}^2)^{1/2}} \qquad (A)$$

if $V_{kk'}$ is known this can be solved to get x_k and hence u_k, v_k.

total number of e^- is

$$N = \sum_k \left(c_k^+ c_k + c_{-k}^+ c_{-k} \right)$$

$$= \sum_k \left[2v_k^2 + (u_k^2 - v_k^2)(m_k + m_{-k}) + 2u_k v_k (\gamma_k^+ \gamma_{-k}^+ + \gamma_{-k} \gamma_k \right.$$

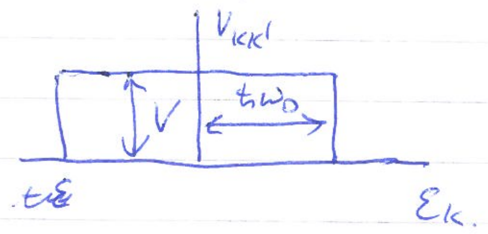
$$= \sum_k 2v_k^2 \qquad (\text{. diagonalized } \& \ m_k = m_{-k} = 0).$$

$$= \sum_k (1 + 2x_k) \qquad (v_k^2 = \tfrac{1}{2} + x_k)$$

when there are no interactions we know.

$$\sum_{k<k_F} 2 = N \quad \Rightarrow \quad x_k = +\tfrac{1}{2} \quad \varepsilon_k < \mu.$$
$$x_k = -\tfrac{1}{2} \quad \varepsilon_k > \mu.$$

Let's take simple form for $V_{kk'}$ again.



(BCS, 1957 article).

$$V_{kk} \begin{cases} V & \text{if } |\hat{\varepsilon}_k| < \hbar\omega_D \\ 0 & \text{otherwise.} \end{cases} \qquad (\hat{\varepsilon}_k = \varepsilon_k - \mu, \text{ i.e. put } \varepsilon_F = 0)$$

$$④: \ \Delta_k = \frac{1}{2} \sum_{k'} V_{kk'} \frac{\Delta_{k'}}{(\varepsilon_{k'}^2 + \Delta_{k'}^2)^{1/2}} = \frac{1}{2} \int_{-\infty}^{\infty} D(\varepsilon_{k'})\, d\varepsilon_{k'} \frac{\Delta_{k'}}{(\varepsilon_{k'}^2 + \Delta_{k'}^2)^{1/2}} V_{kk'}$$

$$= \frac{1}{2} V D(\varepsilon_F) \int_{-\hbar\omega_D}^{\hbar\omega_D} \frac{\Delta}{\sqrt{\varepsilon^2 + \Delta^2}}\, d\varepsilon \qquad (V \text{ constant} \Rightarrow \Delta \text{ constant})$$