Summary.

$$H_{BCS} = H_{BCS}^{kin} + H_{BCS}^{int}$$

$$H_{BCS}^{kin} = \sum_k \varepsilon_k \left[2v_k^2 + (u_k^2 - v_k^2)(m_k + m_{-k}) \right]$$

$$H_{BCS}^{int} = - \sum_{kk'} V_{kk'} \left[u_{k'} v_{k'} u_k v_k (1 - m_{k'} - m_{-k'})(1 - m_k - m_{-k}) \right]$$

with $m_k = \gamma_k^+ \gamma_k$ and $u_k^2 + v_k^2 = 1$.

$\gamma_k = u_k c_k - v_k c_{-k}^+$ and $c_k = u_k \gamma_k + v_k \gamma_{-k}^+$

$\gamma_{-k} = u_k c_{-k} - v_k c_k^+$ $\quad c_{-k} = u_k \gamma_{-k} - v_k \gamma_k^+$

in the above H_{BCS} we already made the cancellation of off-diagonal terms (and neglecting 4th order in γ terms) yielded.

$$\Delta_k = \frac{1}{2} \sum_{k'} V_{kk'} \left(\frac{1}{4} - x_{k'}^2 \right)^{1/2} = \frac{1}{2} \sum_{k'} V_{kk'} \frac{\Delta_{k'}}{(\varepsilon_{k'}^2 + \Delta_{k'}^2)^{1/2}}.$$

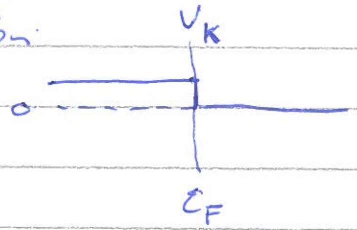
(x_k defined by $u_k = (\frac{1}{2} - x_k)^{1/2}$; $v_k = (\frac{1}{2} + x_k)^{1/2}$

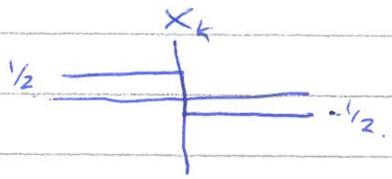
and $x_k = \pm \dfrac{\varepsilon_k}{2(\varepsilon_k^2 + \Delta_k^2)^{1/2}}$.

non-interacting and $T=0 \Rightarrow x_k = 1/2 \quad \varepsilon_k < \mu$
$x_k = -1/2 \quad \varepsilon_k > \mu$)

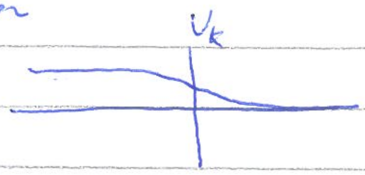
total number of e^- $N = \sum 2 v_k^2 \equiv \sum (1 + 2x_k)$

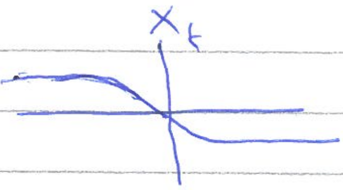
no interaction





with interaction

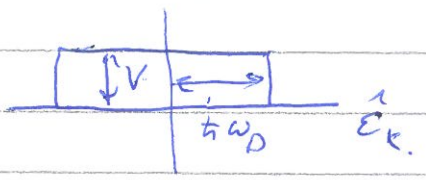




if $\hat{\varepsilon}_f = 0$. (i.e. $\hat{\varepsilon}_F = \hat{\varepsilon}_F - \mu$.) then.

$$x_k = -\frac{\hat{\varepsilon}_k}{2\sqrt{\hat{\varepsilon}_k^2 + \Delta_k^2}}$$

for simple form V_{kk}.



$$V_{kk} = \begin{cases} V & |\varepsilon| < \hbar\omega_D \\ 0 & \text{otherwise} \end{cases}$$

$$\Delta_k = \Delta = \frac{\hbar\omega_D}{\sinh[1/VD(\varepsilon_F)]}.$$

weak coupling $VD(\varepsilon_F) << 1$

$$\Rightarrow \Delta = 2\hbar\omega_D\, e^{-1/VD(\varepsilon_F)}.$$

Ground state energy

$$\mathcal{E}_{S.C.} = \sum_k 2\varepsilon_k^2 v_k^2 - \sum_{kk'} V_{kk'} u_{k'} v_{k'} u_k v_k$$

(i.e. $m_k = m_{-k} = 0$. & off diagonal already out)

$$= \sum_k \varepsilon_k (1+2x_k) - \sum_{kk'} V_{kk'} \left[\left(\tfrac{1}{4} - x_{k'}^2\right)\left(\tfrac{1}{4} - x_k^2\right)\right]^{1/2}$$

using $\Delta_k = \frac{1}{2}\sum_{k'} V_{kk'} \left(\frac{1}{4} - x_{k'}^2\right)^{1/2}$. this becomes.

$$\mathcal{E}_{S.C.} = \sum_k \varepsilon_k (1+2x_k) - \left(\tfrac{1}{4} - x_k^2\right)^{1/2} \Delta$$

normal state non interacting

$$\mathcal{E}_N = \sum_k (1+2x_k)\varepsilon_k = \sum_{k<k_F} 2\varepsilon_k. \qquad \left(x_k = +\tfrac{1}{2} \text{ for } k<k_F,\ -\tfrac{1}{2} \text{ for } k>k_F\right)$$

$$\mathcal{E}_{cond} \equiv \mathcal{E}_{S.C.} - \mathcal{E}_N = \sum_{k<k_F} \varepsilon_k (2x_k - 1) + \sum_{k>k_F} \varepsilon_k (2x_k+1)$$

$$- \sum_k \left(\tfrac{1}{4} - x_k^2\right)^{1/2} \Delta$$

using $x_k = -\dfrac{\varepsilon_k}{2\sqrt{\varepsilon_k^2 + \Delta_k^2}}$ one finds

$$\mathcal{E}_{cond} = 2D(\varepsilon_F) \int_0^{\hbar\omega_D} \left\{ \varepsilon_k - \frac{2\varepsilon_k^2 + \Delta^2}{2\sqrt{\varepsilon_k^2 + \Delta^2}} \right\} d\varepsilon$$

$$= D(\varepsilon_F) \left\{ (\hbar\omega_D)^2 - \hbar\omega_D \left[(\hbar\omega_D)^2 + \Delta^2\right]^{1/2} \right\}.$$

using $\Delta = \hbar\omega_D / \sinh(1/VD(\varepsilon_F))$ this can be written

as $\mathcal{E}_C = (\hbar\omega_D)^2 D(\varepsilon_F) \cdot \left[1 - \coth\left(\frac{1}{VD(\varepsilon_F)}\right)\right]$

which in weak coupling limit becomes.

$$\mathcal{E}_{cond} \simeq -2(\hbar\omega_D)^2 \cdot D(\cancel{\ }\varepsilon_F)\, e^{-2/VD(\varepsilon_F)}$$

$$= -\tfrac{1}{2} D(\varepsilon_F) \cdot \Delta^2 \qquad (\Delta_{\substack{weak\\ coupl.}} = 2\hbar\omega_D e^{-1/VD(\varepsilon_F)})$$

this is about 10^{-7} eV/e^- ($\triangleq$ mK) which is so small since only e^- between $\varepsilon_F - \Delta$ and $\varepsilon_F + \Delta$ contribute.

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ground state wavefunction.

For the ground state we want $m_k = m_{-k} = 0$.
this must mean that.

$$\gamma_k |\psi_0\rangle = \gamma_{-k} |\psi_0\rangle = 0.$$

wave funct. which satisfies this:

$$\left(\prod_k \gamma_k \gamma_{-k} \right) |0\rangle \qquad \text{since } \gamma_k \gamma_k = \gamma_{-k}\gamma_{-k} = 0.$$

written out: $\left(\prod_k \gamma_k \gamma_{-k}\right)|0\rangle =$

$$\left[\prod_k (u_k c_k - v_k c_{-k}^+)(u_k c_{-k} + v_k c_k^+) \right] |0\rangle$$

$$= \left[\prod_k (u_k v_k + v_k^2 c_k^+ c_{-k}^+) \right] |0\rangle$$

divide by $\prod_k v_k$ to normalize

$$\Rightarrow |\psi_0\rangle = \left[\prod_k u_k + v_k c_k^+ c_{-k}^+ \right] |0\rangle.$$

excitations

to create excitations one can operate with

γ_k^+ and γ_{-k}^+ on $|\Psi_0\rangle$

the BCS H-operator can be written as. (see p.17, 19)

$$H_{BCS} = \mathcal{E}_{s.c.} + \sum_k (m_k + m_{-k}) \Big[(u_k^2 - v_k^2)\, \varepsilon_k + 2 u_k v_k \sum_{k'} V_{kk'} u_{k'} v_{k'} \Big]$$

which using solution for u_k, v_k (see p 13) becomes

$$H_{BCS} = \mathcal{E}_{s.c.} + \sum_k (\varepsilon_k^2 + \Delta^2)^{1/2} (m_k + m_{-k}) + \ldots$$

$\Rightarrow$ energy of elementary excitation $E_k = (\varepsilon_k^2 + \Delta^2)^{1/2}$.

however a single γ_k^+ (or γ_{-k}^+) can not create excitation since this only contains single c and c^+ and any excitation involves $\underline{2}$ electrons.

for instance E-field can scatter e^- from k' to k:

$$c_k^+ c_{k'} |\Psi_0\rangle = (u_k \gamma_k^+ + v_k \gamma_{-k}) (u_{k'} \gamma_{k'} + v_{k'} \gamma_{-k'}^+) |\Psi_0\rangle$$

$$= u_k v_{k'} \underbrace{\gamma_k^+ \gamma_{-k'}^+}_{\text{pair of excitations}} |\Psi_0\rangle$$

So any excitation costs at least 2Δ.

$\Rightarrow$ specific heat at low $T \sim e^{-\frac{2\Delta}{kT}}$

optical absorption for $\hbar\omega \geq 2\Delta$

Transition temperature and $\Delta(T)$.

we eliminated the off-diag. terms by determining u_k, v_k using that $T=0$ i.e. $m_k = m_{-k} = 0$.
we'll assume that still works at higher T.
i.e. H_{BCS} is still.

$$H_{BCS} = \sum_k 2\varepsilon_k v_k^2 + \sum_k (u_k^2 - v_k^2)\, \varepsilon_k (m_k + m_{-k}) +$$

$$+ \sum_{kk'} V_{kk'}\, u_{k'} v_{k'} u_k v_k (1 - m_{k'} - m_{-k'})(1 - m_k - m_{-k}).$$

energy to create quasi-particle is then.

$$E_k = \frac{\partial \langle H_{BCS} \rangle}{\partial \langle m_k \rangle} = \varepsilon_k (u_k^2 - v_k^2) + 2 u_k v_k \sum_{k'} V_{kk'} (1 - \langle m_{k'} \rangle - \langle m_{-k'} \rangle)$$

as approximation we could take $\langle m_k \rangle = \langle m_{-k} \rangle = \frac{1}{e^{E_k/kT} + 1}$.

i.e. Fermi-Dirac with $\mu = 0$ while # particles not conserved by γ's.

then to make diagonal terms vanish (see p 12)

one needs that.

$$2\varepsilon_k u_k v_k - (u_k^2 - v_k^2)\sum_{k'} V_{kk'}\, u_{k'} v_{k'} (1 - m_{k'} - m_{-k'}) = 0$$

or

$$2\varepsilon_k u_k v_k - (u_k^2 - v_k)\sum_{k'} V_{kk'}\, u_{k'} v_{k'} (1 - 2f(E_k)) = 0.$$

with $f(E_k) = \dfrac{1}{e^{E_k/kT} + 1}$

if we put (see p 13)

$$\Delta_k(T) = \sum V_{kk'} \left(\tfrac{1}{4} - x_{k'}^2\right)^{1/2} \cdot (1 - 2f(E_k))$$

we have exactly same eq. as before with $\Delta \to \Delta(T)$

$$\Rightarrow \Delta_k(T) = \frac{1}{2}\sum_{k'} V_{kk'} \frac{\Delta_{k'}(T)}{\left[\varepsilon_{k'}^2 + \Delta_{k'}(T)^2\right]^{1/2}} \left\{1 - 2f(E_{k'})\right\}$$

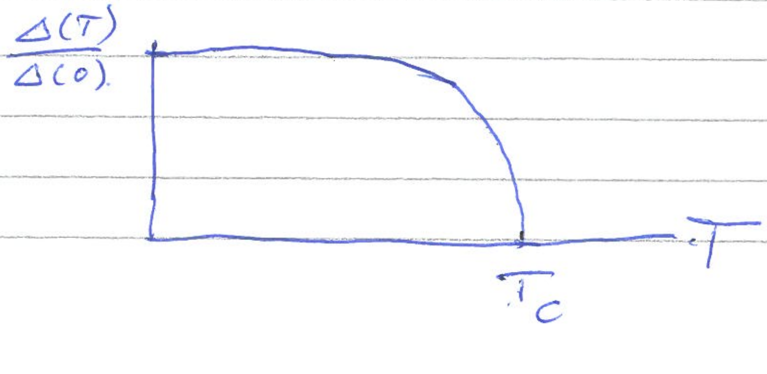
using $E_k = \left[\varepsilon_k^2 + \Delta_k^2(T)\right]^{1/2}$.

this becomes $\displaystyle \Delta_k = \frac{1}{2}\sum_{k'} V_{kk'} \frac{\Delta_{k'}}{(\varepsilon_{k'}^2 + \Delta_{k'}^2)^{1/2}} \tanh \frac{\sqrt{\varepsilon_{k'}^2 + \Delta_{k'}^2}}{2kT}$

using BCS approach for $V_{kk'}$ one gets.

$$V D(\varepsilon_F) \int_0^{\hbar\omega_D} \frac{\tanh\left\{\sqrt{\varepsilon^2 + \Delta^2}/2kT\right\}}{\sqrt{\varepsilon^2 + \Delta^2}}\, d\varepsilon = 1$$

$\Rightarrow$



to find T_c we set $\Delta(T_c)=0$.

~~using the~~ $$VD(\varepsilon_F)\int_0^{\hbar\omega_D}\frac{\tanh \varepsilon/2kT_c}{\varepsilon}\,d\varepsilon = 1.$$

$$VD(\varepsilon_F)\int_0^{\hbar\omega_D/2kT_c}\frac{\tanh x}{x}\,dx = 1.$$

$$\Rightarrow \quad \underline{kT_c = 1.14\,\hbar\omega_D\, e^{-1/VD(\varepsilon_F)}}$$ in weak coupling.

isotope effect

since we already had $\Delta(0) = 2\,\hbar\omega_D\, e^{-1/VD(\varepsilon_F)}$ (see p15)

one has $\dfrac{2\Delta(0)}{kT_c} = 3.50$ $\Rightarrow$ deviations often used as argument non-BCS.