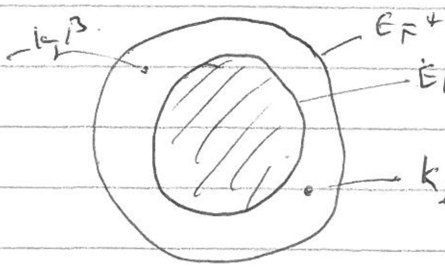
Superconductivity
WS 15/16 11-1-2016.

L.N. Cooper Phys Rev 104 1189 (1956).

Cooper pair problem.



- 2 electrons just above fermi sea
- 2-particles interacting, potential $V(r)$ $r = r_1 - r_2$
- assume singlet
- assume only relative motion relevant. (i.e. center of mass motion zero)

then schrödinger equation;

$$-\frac{\hbar^2}{m}\nabla_r^2\psi + V(r)\psi = E\psi \qquad (1)$$

spatial part of wfunct expanded in terms of plane waves:

$$\psi(\bar{r}) = \frac{1}{\sqrt{V}}\sum_{\bar{k}>\bar{k}_F} g_{\bar{k}}\, e^{i\bar{k}\cdot\bar{r}} \qquad (2).$$

since singlet one must have $\psi(r_1 - r_2) = +\psi(r_2 - r_1)$.

$\Rightarrow \quad g_k = g_{-k}$

substituting (2) into (1)

$$-\frac{\hbar^2}{m}\nabla^2\frac{1}{\sqrt{V}}\sum_{k'} g_{k'}\, e^{ik'r} + \frac{1}{\sqrt{V}}\sum_{k'} V(r)\, e^{ik'r} g_{k'} = \frac{1}{\sqrt{V}} E \sum_{k'} g_{k'} e^{ik'r}$$

$$\cancel{\frac{1}{\sqrt{V}}}\sum_{k'}\frac{\hbar^2 k'^2}{m} g_{k'}\, e^{ik'r} + \cancel{\frac{1}{\sqrt{V}}}\sum_{k'} V(r)\, e^{ik'r} g_{k'} = \cancel{\frac{1}{\sqrt{V}}} E \sum_{k'} g_{k'} e^{ik'r}$$

or $$\sum_{k'}\left(-\frac{\hbar^2 k'^2}{m} + E\right) g_{k'}\, e^{ik'r} = \sum_{k'} V(r)\, e^{ik'r}.$$

Left multiply by e^{-ikr} ~~gives~~ and integrating over volume gives

$$\sum_{k'} g_{k'} \{ E - 2\varepsilon_k \} \int e^{-i(k-k')r} d^3r = \sum_{k'} g_{k'} \int e^{-ikr} V(r) e^{ik'r} d^3r$$

def.: $V_{kk'} = \frac{1}{V} \int e^{-i(k-k')r} V(r)\, d^3r$

note that $\frac{1}{V} \int e^{-i(k-k')r} d^3r = \delta_{kk'}$

then

$$\sum_{k'} g_{k'} \{ E - 2\varepsilon_k \} \delta_{kk'} = \sum_{k'} g_{k'} V_{kk'}$$

or $$g_k \{ E - 2\varepsilon_k \} = \sum_{k'} g_{k'} V_{kk'} \qquad (3)$$

~~[illegible]~~ to further proceed one needs to know $V(r)$ or $V_{kk'}$

assumption: $V_{kk'} = \begin{cases} -V_0 & \varepsilon_F < \varepsilon < \varepsilon_F + \hbar\omega_D \\ 0 & \text{elsewhere.} \end{cases}$

(3) then becomes $g_k = \frac{-V_0}{E - 2\varepsilon_k} \sum_{k'} g_{k'}$

or $$\sum_k g_k = \sum_k \frac{-V_0}{E - 2\varepsilon_k} \sum g_{k'}$$

$$\boxed{1 = \sum_k \frac{V_0}{2\varepsilon_k - E}}$$

changing summation to integration over shell $\varepsilon_F \to \varepsilon_F + \hbar\omega_D$.

$$1 = V_0 \int_{\varepsilon_F}^{\varepsilon_F + \hbar\omega_D} \frac{N(\varepsilon)}{2\varepsilon - E}\, d\varepsilon.$$

$N(\varepsilon)$: D.O.S. for one spin

assume $N(\varepsilon)$ constant $= N(\varepsilon_F)$.

$$1 = V_0 N(\varepsilon_F) \cdot \int_{\varepsilon_F}^{\varepsilon_F + \hbar\omega_D} \frac{1}{2\varepsilon - E}\, d\varepsilon$$

$$1 = V_0 N(\varepsilon_F) \frac{1}{2} \int_{\varepsilon_F}^{\varepsilon_F + \hbar\omega_D} \frac{1}{2\varepsilon - E}\, d(2\varepsilon - E)$$

$$1 = \frac{V_0 N(\varepsilon_F)}{2} \cdot \ln(2\varepsilon - E) \Big|_{\varepsilon_F}^{\varepsilon_F + \hbar\omega_D}$$

$$1 = \frac{V_0 N(\varepsilon_F)}{2} \cdot \ln\left[\frac{2(\varepsilon_F + \hbar\omega_D) - E}{2\varepsilon_F - E}\right]$$

$$e^{-\frac{2}{V_0 N(\varepsilon_F)}} = \frac{2(\varepsilon_F + \hbar\omega_D) - E}{2\varepsilon_F - E}$$

solve for E : $E = 2\varepsilon_F - \dfrac{2\hbar\omega_D \cdot e^{-\frac{2}{V_0 N(\varepsilon_F)}}}{1 - e^{-2/V_0 N(\varepsilon_F)}}$.

$\Rightarrow$ bound state with $E < 2\varepsilon_F$ for $\underline{\underline{\text{all}}}$ $V_0 N(\varepsilon_F) > 0$!

typically $V_0 N(\varepsilon_F) < 0.3$ (weak coupling).

$$\Rightarrow E = 2\varepsilon_F - 2\hbar\omega_D e^{-\frac{2}{V_0 N(\varepsilon_F)}}$$

Size of cooper pair.

calculate mean square distance between electrons:

$$\overline{r^2} = \frac{\int |\psi|^2 r^2 d^3r}{\int |\psi|^2 d^3 r}. \qquad \text{with } \psi(r) = \frac{1}{\sqrt{V}} \sum g_k e^{ikr}$$

$$|\psi|^2 = \frac{1}{V} \sum_{kk'} g_k g^*_{k'} e^{i(k-k')r}$$

$$\int |\psi|^2 d^3 r = \sum_{kk'} g_k g^*_{k'} \delta_{kk'} = \sum_k |g_k|^2.$$

$$\int |\psi|^2 r^2 d^3 r : \quad -i \nabla_k e^{ikr} = r e^{ikr}.$$

$$\Rightarrow \int |\psi|^2 r^2 d^3 r = \sum_k |\nabla_k g_k|^2.$$

so that $$\overline{r^2} = \frac{\sum |\nabla_k g_k|^2}{\sum |g_k|^2}$$

we know $$g_k \sim \frac{1}{2\varepsilon_k - E}$$

$$\frac{\partial}{\partial k} = \frac{\partial}{\partial \varepsilon_k} \frac{\partial \varepsilon_k}{\partial k} = \hbar v_F \frac{\partial}{\partial \varepsilon_k}.$$

change $\sum \to \int$

so $$\overline{r^2} = \frac{(\hbar v_F)^2 \cdot \int \left(\frac{\partial}{\partial \varepsilon} \cdot \frac{1}{2\varepsilon - E} \right)^2 d\varepsilon}{\int \left(\frac{1}{2\varepsilon - E} \right)^2 d\varepsilon}$$

$$= \frac{-\frac{2}{3} (\hbar v_F)^2 \cdot \left(\frac{1}{2\varepsilon - E} \right)^3 \Big|_0^\infty}{-\frac{1}{2} \frac{1}{2\varepsilon - E} \Big|_0^\infty} = \frac{4}{3} \frac{(\hbar v_F)^2}{E^2}$$

to get an idea: take $\varepsilon \sim k_B T_c$, $T_c = 10K$.
$v_F = 10^8 cm/s$

$$\Rightarrow \sqrt{\overline{r^2}} \simeq 10^{-4} \text{ cm} = 10^4 \text{ Å}$$

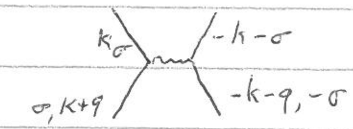
in a metal an electron occupies typically $(2\text{Å})^3$.
so within the cooperpair $\sim \frac{(10^4)^3}{(2)^3} \approx 10^{11}$ other e^-

Cooper pair in 2nd quant.

wfnct $2e^-$: $|\Psi_{12}\rangle = \sum_{k_1 k_2 \sigma_1 \sigma_2} g_{k_1 k_2 \sigma_1 \sigma_2} c^\dagger_{k_1\sigma_1} c^\dagger_{k_2\sigma_2} |FS\rangle$

cooperpair: $k_1 = -k_2$; $\sigma_1 = -\sigma_2$.

$$\Rightarrow |\Psi_{12}\rangle = \sum_k g_k c^\dagger_{k\sigma} c^\dagger_{-k-\sigma} |FS\rangle$$



Hamilton operator.

(in principle V_q)

$$H = \sum_{k\sigma} \varepsilon_k c^\dagger_{k\sigma} c_{k\sigma} - \frac{V}{2} \sum_{\sigma k q} c^\dagger_{k+q\,\sigma} c^\dagger_{-k-q\,-\sigma} c_{-k-\sigma} c_{k\sigma}$$

kin.E term: $\langle \Psi_{12} | H_{kin} | \Psi_{12} \rangle = \langle FS | \sum_l g^*_l c_{l\sigma} c_{-l-\sigma} \cdot \sum_{k\sigma} \varepsilon_k c^\dagger_{k\sigma} c_{k\sigma} \cdot \sum_m g_m c^\dagger_{m\sigma} c^\dagger_{-m-\sigma} |FS\rangle$

only term which survives: $l = m = k$; σ: 2 values.

$$\langle \Psi_{12} | H_{kin} | \Psi_{12} \rangle = 2 \sum_k \varepsilon_k |g_k|^2 .$$

interaction part.

$$-\frac{V}{2}\langle FS|\sum_{l} g_l^* c_{l\sigma} c_{-l-\sigma} \cdot \sum_{kq\sigma} c^+_{k+q\sigma} c^+_{-k-q-\sigma} c_{-k-\sigma} c_{k\sigma} \cdot \sum_m g_m c^+_{m\sigma} c^+_{-m-\sigma} |FS\rangle$$

gives. $-V\sum_{k,q} g^*_{k+q}\, g_k.$

So energy is

$$E = 2\sum_k \varepsilon_k |g_k|^2 - V\sum_{kq} g^*_{k+q}\, g_k$$

normal. constraint: $\sum_k |g_k|^2 = 1.$

use lagrange mult. λ to put constraint in energy.

$$E' = E - \lambda \sum_k |g_k|^2. \quad \text{and minimize for } g_k^*.$$

$$\frac{\partial E'}{\partial g_k^*} = \frac{\partial}{\partial g_k^*}\left[2\sum_k \varepsilon_k |g_k|^2 - V\sum_{k,q} g^*_{k+q} g_k - \lambda\sum_k |g_k|^2\right]$$

$$= 2\varepsilon_k g_k - \underbrace{V\sum_q g_{k-q}}_{=\sum_k g_k.} - \lambda g_k = 0$$

$$\Rightarrow (2\varepsilon_k - \lambda) g_k = V\sum_k g_k$$

def. $\sum_k g_k \equiv C$

$$(2\varepsilon_k - \lambda) g_k = VC \quad \text{or} \quad g_k = \frac{VC}{(2\varepsilon_k - \lambda)}$$

as before sum over k.

$$\sum_k g_k = \sum_k \frac{VC}{2\varepsilon_k - \lambda} \quad \Rightarrow \quad 1 = \sum_k \frac{V}{2\varepsilon_k - \lambda}$$